

GREEN'S FUNCTIONS FOR PARABOLIC SYSTEMS OF SECOND ORDER IN TIME-VARYING DOMAINS

HONGJIE DONG AND SEICK KIM

ABSTRACT. We construct Green's functions for divergence form, second order parabolic systems in non-smooth time-varying domains whose boundaries are locally represented as graph of functions that are Lipschitz continuous in the spatial variables and $1/2$ -Hölder continuous in the time variable, under the assumption that weak solutions of the system satisfy an interior Hölder continuity estimate. We also derive global pointwise estimates for Green's function in such time-varying domains under the assumption that weak solutions of the system vanishing on a portion of the boundary satisfy a certain local boundedness estimate and a local Hölder continuity estimate. In particular, our results apply to complex perturbations of a single real equation.

1. INTRODUCTION

Green's functions play an important role in the solution of elliptic and parabolic partial differential equations. There is a large literature on Green's functions of uniformly elliptic and parabolic equations in divergence form. Green's functions of elliptic equations of divergence form with L_∞ coefficients have been extensively studied by Littman et al. [23] and Grüter and Widman [12]; see also [7, 10, 11]. Recently, Hofmann and Kim [14] gave a unified approach in studying Green's functions for both scalar equations and systems of elliptic type; see also [8]. For parabolic equations, Aronson [1] established two-sided Gaussian bounds for the fundamental solutions of parabolic equations in divergence form with L_∞ coefficients; see also [2, 6, 9, 13, 18, 28] and references therein for related results. In a recent paper by Cho and the authors [4], we proved that if weak solutions of a given parabolic system satisfy an interior Hölder continuity estimate, then the Green's function of the system exists in any cylindrical domain. In the scalar case, such an interior Hölder continuity estimate is a consequence of Nash [25] and Moser [24], and also such an estimate is available for weak solutions of a system if, for example, its principal coefficients are VMO in the spatial variables. However, the construction of Green's function in [4] heavily relied on the results by Ladyzhenskaya and Ural'tseva that are available only for cylindrical domains. In another recent article by Cho and the authors [5], we demonstrated how one can derive global pointwise estimates for the Green's function in a cylindrical domain by using a local boundedness estimate and a local Hölder estimate for the weak solutions of the parabolic system vanishing on a portion of the boundary.

The aim of this article is to give results similar to those of [4, 5] for a class non-smooth time-varying domains whose boundaries are given locally as graph of functions that are Lipschitz continuous in the spatial variables and $1/2$ -Hölder continuous in the time variable, which hereafter shall be referred to as time-varying H_1 domains. There are many papers dealing with parabolic equations in this type of time-varying domains. Lewis and

2000 *Mathematics Subject Classification.* Primary 35A08, 35K40; Secondary 35B45.

Key words and phrases. Green's function; Green's matrix; parabolic system; time-varying domain.

Murray [21] considered a domain in $\mathbb{R} \times \mathbb{R}^d$ of the form $\{(t, x) : x^d > f(t, x')\}$, where $x' = (x^1, \dots, x^{d-1})$ and f is a function that is Lipschitz in x' and whose t -derivative of order $1/2$ belongs to BMO, which is slightly stronger than $f \in H_1$. The non-cylindrical domains considered by Hofmann and Lewis in their important paper [15] on L_2 boundary value problems for the heat equation are also included in time-varying H_1 domains; see also [16, 27, 29]. Brown et al. [3] investigated weak solutions of parabolic equations in time-varying H_1 domains and proved the unique solvability of Dirichlet boundary value problems. We shall in fact utilize their result in constructing the Green's function in time-varying H_1 domains. In contrast, there is little literature on Green's functions of parabolic equations in non-cylindrical domains and to the best of our knowledge, there is no literature dealing with Green's function for parabolic systems of second order with L_∞ coefficients in time-varying H_1 domains; see the remarks made in the last paragraph of the introduction.

We denote by $\mathbf{u} = (u^1, \dots, u^m)^T$ a vector-valued function of $d+1$ independent variables $(t, x^1, \dots, x^d) = (t, x) = X$. We consider parabolic systems of second-order

$$(1.1) \quad \mathcal{L}\mathbf{u} := \mathbf{u}_t - D_\alpha(A^{\alpha\beta}D_\beta\mathbf{u}),$$

where the usual summation conventions are assumed and $A^{\alpha\beta} = A^{\alpha\beta}(X)$, for $\alpha, \beta = 1, \dots, d$, are $m \times m$ matrices whose entries are L_∞ functions satisfying the strong ellipticity condition; see Section 2.2 for the details. We emphasize that the coefficients are not assumed to be time independent or symmetric. We will later impose some further assumptions on the operator \mathcal{L} but not explicitly on its coefficients. By a Green's function for the system (1.1) in a time-varying H_1 domain Ω we mean an $m \times m$ matrix valued function $\mathbf{G}(X, Y) = \mathbf{G}(t, x, s, y)$ which satisfies the following for all $Y \in \Omega$:

$$\begin{aligned} \mathcal{L}\mathbf{G}(\cdot, Y) &= \delta_Y(\cdot)I_m \text{ in } \Omega, \\ \mathbf{G}(\cdot, Y) &= 0 \text{ on } \mathcal{P}\Omega, \end{aligned}$$

where $\delta_Y(\cdot)$ is a Dirac delta function, I_m is $m \times m$ identity matrix, and $\mathcal{P}\Omega$ denotes the parabolic boundary of Ω ; see Section 2.6 for more precise definition. In this article, we prove that if weak solutions of (1.1) satisfy an interior Hölder continuity estimate, then there exists a unique Green's function in Ω and it satisfies some natural growth properties; see Theorem 3.1 below. Moreover, we show that the Green's function also satisfies the following familiar property:

$$\lim_{t \rightarrow s_+} \mathbf{G}(t, x, s, \cdot) = \delta_x(\cdot)I_m \text{ on } \omega(s) = \{y \in \mathbb{R}^d : (s, y) \in \Omega\}.$$

We also derive the following global Gaussian estimate for the Green's function in a time-varying H_1 domain by using a local boundedness estimate for the weak solutions of (1.1) vanishing on a portion of the boundary: For any $T > 0$, there exists $N > 0$ such that for all $X = (t, x)$ and $Y = (s, y)$ in Ω satisfying $0 < t - s < T$, we have

$$(1.2) \quad |\mathbf{G}(t, x, s, y)| \leq \frac{N}{(t-s)^{d/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\},$$

where $\kappa > 0$ is a constant independent of T ; see Theorem 3.11 and Remark 3.13. In particular, the above estimate (1.2) holds in the scalar case (i.e., when $m = 1$) and also in the case of L_∞ -perturbation of diagonal systems; see Corollary 4.1 and Section 4.2 below. In fact, in such cases, a stronger estimate is available near the boundary. For any $T > 0$, there exists $N > 0$ such that for all $X = (t, x)$ and $Y = (s, y)$ in Ω satisfying $0 < t - s < T$, we have

$$(1.3) \quad |\mathbf{G}(t, x, s, y)| \leq N \left(1 \wedge \frac{d(X)}{|X-Y|_{\mathcal{P}}}\right)^\mu \left(1 \wedge \frac{d(Y)}{|X-Y|_{\mathcal{P}}}\right)^\mu \frac{1}{(t-s)^{d/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\},$$

where $\kappa > 0$ and $\mu \in (0, 1]$ are constants independent of T , and we used the notation $a \wedge b = \min(a, b)$, $|X - Y|_{\mathcal{D}} = \max(\sqrt{|t - s|}, |x - y|)$, and $d(X) = \inf\{|Z - X|_{\mathcal{D}} : Z \in \partial\Omega\}$. We show how to derive a global estimate like (1.3) for the Green's function in a time-varying H_1 domain by using a local Hölder continuity estimate for the weak solutions of (1.1) vanishing on a portion of the boundary; see Theorem 3.16 and Remark 3.19. As mentioned above, the estimate (1.3) particularly holds in the case of L_∞ -perturbation of diagonal system as well as in the scalar case; see Corollary 4.4 and Section 4.2.

The organization of the paper is as follows. In Section 2, we introduce some notation and definitions including the precise definitions of time-varying H_1 domains and Green's functions of the system (1.1) in such domains. In Section 3, we state our main theorems and give a few remarks concerning extensions of them. In Section 4, we present some applications of our main results including applications to the scalar case, L_∞ -perturbation of diagonal systems, and systems with VMO_x coefficients. We provide proofs of our main theorems in Section 5 and some technical lemmas are proved in the appendix.

Finally, several remarks are in order. In the scalar case, there are a few papers discussing Green's functions in non-cylindrical domains. However, we believe that even in the scalar case, our results give still new perspectives on Green's functions. In [26], Nyström constructed Green's functions in bounded time-varying H_1 domains utilizing the fundamental solutions and the caloric measures, and in doing so, he made a qualitative assumption that the coefficients are smooth in order to have well-defined concept of solutions; i.e. to assume that all solutions are classical ones. The main drawback of this kind of approach is that it is not well suited to handle unbounded domains, especially domains with unbounded cross-sections such as the graph domains considered by Hofmann and Lewis [15]. The novelty of our paper lies in presenting a powerful unifying method that establishes the existence and various estimates for the Green's function of parabolic equations and systems with L_∞ coefficients in time-varying H_1 domains including the graph domains. Also, even though we impose some conditions on the operator \mathcal{L} in the vectorial case, we do not make any smoothness assumption on its coefficients in order to assume that the solutions of the system are classical. Moreover, the treatment of L_∞ -perturbation of diagonal systems is a unique feature of our paper and we believe that it could find some interesting applications in the complex perturbation theory for the Dirichlet problem of second order parabolic equations in time-varying domains.

2. NOTATION AND DEFINITIONS

2.1. Basic notation. We mostly follow notation employed in Ladyzhenskaya et al. [20], supplemented by that used in Lieberman [22]. First we use $X = (t, x) = (t, x^1, \dots, x^d)$ to denote a point in \mathbb{R}^{d+1} with $d \geq 1$ and we denote $X' = (t, x') = (t, x^1, \dots, x^{d-1}) \in \mathbb{R}^d$ so that $X = (X', x^d)$. We also write $Y = (s, y) = (s, y', y^d) = (Y', y^d)$. We denote

$$a \wedge b = \min(a, b), \quad a \vee b = \max(a, b) \quad \text{for } a, b \in [-\infty, \infty].$$

We define the parabolic distance in \mathbb{R}^{d+1} and \mathbb{R}^d , respectively, by

$$|X - Y|_{\mathcal{D}} = \sqrt{|t - s|} \vee |x - y|, \quad |X' - Y'|_{\mathcal{D}} = \sqrt{|t - s|} \vee |x' - y'|,$$

where $|\cdot|$ denotes the usual Euclidean norm, and write $|X|_{\mathcal{D}} = |X - 0|_{\mathcal{D}}$. We define the parabolic Hölder norm as follows:

$$|u|_{\mu/2, \mu; \Omega} = [u]_{\mu/2, \mu; \Omega} + |u|_{0; \Omega} := \sup_{\substack{X, Y \in \Omega \\ X \neq Y}} \frac{|u(X) - u(Y)|}{|X - Y|_{\mathcal{D}}^\mu} + \sup_{X \in \Omega} |u(X)|, \quad \mu \in (0, 1].$$

By $C^{\mu/2,\mu}(\Omega)$ we denote the set of all bounded measurable functions u on Ω for which $|u|_{\mu/2,\mu;\Omega}$ is finite. We write $D_i u = D_{x^i} u = \partial u / \partial x^i$ and $u_t = \partial u / \partial t$. We also write $Du = D_x u$ for the vector $(D_1 u, \dots, D_d u)$. We use the following notation for basic cylinders in \mathbb{R}^{d+1} :

$$\begin{aligned} Q(X_0, R) &= \{X \in \mathbb{R}^{d+1} : |X - X_0|_{\mathcal{P}} < R\}, \\ Q_-(X_0, R) &= \{X = (t, x) \in \mathbb{R}^{d+1} : |X - X_0|_{\mathcal{P}} < R, t < t_0\}, \\ Q_+(X_0, R) &= \{X = (t, x) \in \mathbb{R}^{d+1} : |X - X_0|_{\mathcal{P}} < R, t > t_0\}. \end{aligned}$$

We also use the ball $B(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$. For convenience, the parameter X_0 (or x_0) in the notation above is omitted if $X_0 = 0$ (or $x_0 = 0$, respectively). We use Ω to denote a domain (open connected set) in \mathbb{R}^{d+1} . For a fixed number t_0 , we write $\omega(t_0)$ for the set of all points (t_0, x) in Ω , and write $I(\Omega)$ for the set of all t such that $\omega(t)$ is not empty. For $-\infty \leq t_0 < t_1 \leq \infty$, we denote

$$\Omega(t_0, t_1) = \{X = (t, x) \in \Omega : t_0 < t < t_1\}.$$

The parabolic boundary $\mathcal{P}\Omega$ is defined to be the set of all points $X_0 \in \partial\Omega$ such that for any $\varepsilon > 0$, the cylinder $Q_-(X_0, \varepsilon)$ contains points not in Ω . We define $B\Omega$ to be the set of all points $X_0 \in \mathcal{P}\Omega$ such that there is a positive R with $\widetilde{Q}_+(X_0, R) \subset \Omega$ and $S\Omega = \mathcal{P}\Omega \setminus B\Omega$. We define the “time-reversed” parabolic boundary $\widetilde{\mathcal{P}}\Omega$ to be the set of all points $X_0 \in \partial\Omega$ such that for any $\varepsilon > 0$, the cylinder $Q_+(X_0, \varepsilon)$ contains points not in Ω . We also define

$$\Omega[X, R] = \Omega \cap Q(X, R), \quad \mathcal{P}\Omega[X, R] = \mathcal{P}\Omega \cap Q(X, R), \quad \widetilde{\mathcal{P}}\Omega[X, R] = \widetilde{\mathcal{P}}\Omega \cap Q(X, R),$$

and similarly $\Omega_{\pm}[X, R]$, $\mathcal{P}\Omega_{\pm}[X, R]$, and $\widetilde{\mathcal{P}}\Omega_{\pm}[X, R]$. Finally, we define distance functions

$$\begin{aligned} d_{\Omega}(X) &= d(X) = \inf\{|Y - X|_{\mathcal{P}} : Y \in \partial\Omega\}, \\ d_{\Omega}^-(X) &= d^-(X) = \inf\{|Y - X|_{\mathcal{P}} : Y \in \mathcal{P}\Omega, s \leq t\}, \\ d_{\Omega}^+(X) &= d^+(X) = \inf\{|Y - X|_{\mathcal{P}} : Y \in \widetilde{\mathcal{P}}\Omega, s \geq t\}. \end{aligned}$$

2.2. Strongly parabolic systems. Let the operator \mathcal{L} be defined as in (1.1). We assume that the coefficient of \mathcal{L} are defined in the whole space \mathbb{R}^{d+1} in a measurable way and that the principal coefficients $A^{\alpha\beta}$ with the components $A_{ij}^{\alpha\beta}$ satisfy the strong ellipticity

$$(2.1) \quad \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^d A_{ij}^{\alpha\beta}(X) \xi_{\beta}^j \xi_{\alpha}^i \geq \nu \sum_{i=1}^m \sum_{\alpha=1}^d |\xi_{\alpha}^i|^2 =: \nu |\xi|^2, \quad \forall \xi \in \mathbb{R}^{dm}, \quad \forall X \in \mathbb{R}^{d+1},$$

and the uniform boundedness condition

$$(2.2) \quad \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^d |A_{ij}^{\alpha\beta}(X)|^2 \leq \nu^{-2}, \quad \forall X \in \mathbb{R}^{d+1},$$

for some constant $\nu \in (0, 1]$. The adjoint operator ${}^t\mathcal{L}$ is defined by

$${}^t\mathcal{L}u = -u_t - D_{\alpha}(\widetilde{A}^{\alpha\beta} D_{\beta} u),$$

where $\widetilde{A}^{\alpha\beta} = (A^{\beta\alpha})^T$; i.e., $\widetilde{A}_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. Notice that the coefficients $\widetilde{A}_{ij}^{\alpha\beta}$ satisfy the conditions (2.1) and (2.2) with the same constant ν .

2.3. Time-varying H_1 domain. We shall say that Ω is a time-varying H_1 graph domain in \mathbb{R}^{d+1} if there is a function $f = f(X') = f(t, x')$ from \mathbb{R}^d to \mathbb{R} satisfying

$$(2.3) \quad |f(X') - f(Y')| \leq M|X' - Y'|, \quad \forall X', Y' \in \mathbb{R}^d,$$

for some constant $M > 0$ so that Ω is represented by

$$\Omega = \{X = (X', x^d) \in \mathbb{R}^{d+1} : x^d > f(X')\}.$$

We shall say that Ω is a time-varying H_1 domain in \mathbb{R}^{d+1} if

- i) $I(\Omega) = \mathbb{R}$ and $\omega(t)$ is a bounded domain in \mathbb{R}^d for all $t \in \mathbb{R}$.
- ii) There are constants M and $R_a > 0$ such that for each $X_0 \in \partial\Omega$, there is a function $f = f(X') = f(t, x')$ satisfying (2.3), for which (after a suitable rotation of x -axes)

$$\Omega \cap Q(X_0, R_a) = \{X \in Q(X_0, R_a) : x^d > f(X')\}.$$

2.4. Function spaces. For $q \geq 1$, we let $L_q(\Omega)$ denote the classical Banach space consisting of measurable functions on Ω that are q -integrable. The space $W_q^{0,1}(\Omega)$ denotes the set of functions $u \in L_q(\Omega)$ with its weak derivative $Du \in L_q(\Omega)$ having a finite norm

$$\|u\|_{W_q^{0,1}(\Omega)} = \|u\|_{L_q(\Omega)} + \|Du\|_{L_q(\Omega)}.$$

We denote by $W_2^{1,1}(\Omega)$ the Hilbert space with the inner product

$$\langle u, v \rangle_{W_2^{1,1}(\Omega)} := \int_{\Omega} uv + \sum_{\alpha=1}^d \int_{\Omega} D_{\alpha} u D_{\alpha} v + \int_{\Omega} u_t v_t.$$

We define $V_2(\Omega)$ as the set of all $u \in W_2^{0,1}(\Omega)$ having a finite norm $\|u\|_{V_2(\Omega)}$ defined by

$$\|u\|_{V_2(\Omega)}^2 := \int_{\Omega} |Du|^2 dX + \text{ess sup}_{t \in I(\Omega)} \int_{\omega(t)} u^2 dx.$$

The space $V_2^{0,1}(\Omega)$ is obtained by completing the set $W_2^{1,1}(\Omega)$ in the norm of $V_2(\Omega)$. Let $\Sigma \subset \overline{\Omega}$ and u be a $V_2^{0,1}(\Omega)$ function. We say that u vanishes (or write $u = 0$) on Σ if u is a limit in $V_2^{0,1}(\Omega)$ of a sequence of functions in $C_c^{\infty}(\overline{\Omega} \setminus \Sigma)$. We define $\dot{V}_2^{0,1}(\Omega)$ to be the set of all functions u in $V_2^{0,1}(\Omega)$ that vanishes on $S\Omega$.

2.5. Weak Solutions. For $f, g_{\alpha} \in L_{1,loc}(\Omega)^m$ ($\alpha = 1, \dots, d$), we say that u is a weak solution of $\mathcal{L}u = f + D_{\alpha} g_{\alpha}$ in Ω if $u \in V_2(\Omega)^m$ and satisfies

$$(2.4) \quad - \int_{\Omega} u^i \phi_t^i + \int_{\Omega} A_{ij}^{\alpha\beta} D_{\beta} u^j D_{\alpha} \phi^i = \int_{\Omega} f^i \phi^i - \int_{\Omega} g_{\alpha}^i D_{\alpha} \phi^i, \quad \forall \phi \in C_c^{\infty}(\Omega)^m.$$

We say that u is a weak solution of ${}^t\mathcal{L}u = f + D_{\alpha} g_{\alpha}$ in Ω if $u \in V_2(\Omega)^m$ and satisfies

$$(2.5) \quad \int_{\Omega} u^i \phi_t^i + \int_{\Omega} \tilde{A}_{ij}^{\alpha\beta} D_{\beta} u^j D_{\alpha} \phi^i = \int_{\Omega} f^i \phi^i - \int_{\Omega} g_{\alpha}^i D_{\alpha} \phi^i, \quad \forall \phi \in C_c^{\infty}(\Omega)^m.$$

For $\psi_0 = \psi_0(x) \in L_{1,loc}(\omega(t_0))^m$, we say that u is a weak solution of the problem

$$\mathcal{L}u = f + D_{\alpha} g_{\alpha} \text{ in } \Omega(t_0, t_1), \quad u = 0 \text{ on } S\Omega(t_0, t_1), \quad u = \psi_0 \text{ on } \omega(t_0)$$

if $u \in \dot{V}_2^{0,1}(\Omega(t_0, t_1))$ and satisfies for all $\tau \in I(\Omega(t_0, t_1))$ the identity

$$\begin{aligned} \int_{\omega(\tau)} u^i \phi^i dx - \int_{\Omega(t_0, \tau)} u^i \phi_t^i dX + \int_{\Omega(t_0, \tau)} A_{ij}^{\alpha\beta} D_{\beta} u^j D_{\alpha} \phi^i dX &= \int_{\Omega(t_0, \tau)} f^i \phi^i dX \\ - \int_{\Omega(t_0, \tau)} g_{\alpha}^i D_{\alpha} \phi^i dX + \int_{\omega(t_0)} \psi_0^i \phi^i dx, \quad \forall \phi \in C_c^{\infty}(\overline{\Omega(t_0, t_1)} \setminus S\Omega(t_0, t_1))^m. \end{aligned}$$

Similarly, for $\psi_0 = \psi_0(x) \in L_{1,loc}(\omega(t_1))^m$, we say that \mathbf{u} is a weak solution of the problem

$${}^t\mathcal{L}\mathbf{u} = \mathbf{f} + D_\alpha \mathbf{g}_\alpha \text{ in } \Omega(t_0, t_1), \quad \mathbf{u} = 0 \text{ on } S\Omega(t_0, t_1), \quad \mathbf{u} = \psi_0 \text{ on } \omega(t_1)$$

if $\mathbf{u} \in \dot{V}_2^{0,1}(\Omega(t_0, t_1))$ and satisfies for all $\tau \in I(\Omega(t_0, t_1))$ the identity

$$\begin{aligned} \int_{\omega(\tau)} u^i \phi^i dx + \int_{\Omega(\tau, t_1)} u^i \phi_t^i dX + \int_{\Omega(\tau, t_1)} \tilde{A}_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i dX &= \int_{\Omega(\tau, t_1)} f^i \phi^i dX \\ - \int_{\Omega(\tau, t_1)} g_\alpha^i D_\alpha \phi^i dX + \int_{\omega(t_1)} \psi_0^i \phi^i dx, \quad \forall \phi \in C_c^\infty(\overline{\Omega(t_0, t_1)} \setminus S\Omega(t_0, t_1))^m. \end{aligned}$$

2.6. Green's function. Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . We say that an $m \times m$ matrix valued function $\mathbf{G}(X, Y) = \mathbf{G}(t, x, s, y)$, with entries $G_{ij}(X, Y)$ defined on the set $\{(X, Y) \in \Omega \times \Omega : X \neq Y\}$, is a Green's function of \mathcal{L} in Ω if it satisfies the following properties:

i) $\mathbf{G}(\cdot, Y) \in W_{1,loc}^{0,1}(\Omega)$ and $\mathcal{L}\mathbf{G}(\cdot, Y) = \delta_Y I_m$ for all $Y \in \Omega$, in the sense that

$$\int_{\Omega} (-G_{ik}(\cdot, Y) \phi_t^i + A_{ij}^{\alpha\beta} D_\beta G_{jk}(\cdot, Y) D_\alpha \phi^i) = \phi^k(Y), \quad \forall \phi \in C_c^\infty(\Omega)^m.$$

ii) $\mathbf{G}(\cdot, Y) \in V_2^{0,1}(\Omega \setminus Q(Y, R))$ for all $Y \in \Omega$ and $R > 0$, and $\mathbf{G}(\cdot, Y)$ vanishes on $S\Omega$.

iii) For any $\mathbf{f} = (f^1, \dots, f^m)^T \in C_c^\infty(\Omega)$, the function \mathbf{u} given by

$$\mathbf{u}(X) := \int_{\Omega} \mathbf{G}(Y, X) \mathbf{f}(Y) dY$$

belongs to $\dot{V}_2^{0,1}(\Omega)$ and satisfies ${}^t\mathcal{L}\mathbf{u} = \mathbf{f}$ in the sense of (2.5).

Similarly, we say that an $m \times m$ matrix valued function $\tilde{\mathbf{G}}(X, Y) = \tilde{\mathbf{G}}(t, x, s, y)$ is a Green's function of $\tilde{\mathcal{L}}$ in Ω if it satisfies the following properties:

i) $\tilde{\mathbf{G}}(\cdot, Y) \in W_{1,loc}^{0,1}(\Omega)$ and $\tilde{\mathcal{L}}\tilde{\mathbf{G}}(\cdot, Y) = \delta_Y I_m$ for all $Y \in \Omega$, in the sense that

$$\int_{\Omega} (\tilde{G}_{ik}(\cdot, Y) \phi_t^i + \tilde{A}_{ij}^{\alpha\beta} D_\beta \tilde{G}_{jk}(\cdot, Y) D_\alpha \phi^i) = \phi^k(Y), \quad \forall \phi \in C_c^\infty(\Omega)^m.$$

ii) $\tilde{\mathbf{G}}(\cdot, Y) \in V_2^{0,1}(\Omega \setminus Q(Y, R))$ for all $Y \in \Omega$ and $R > 0$, and $\tilde{\mathbf{G}}(\cdot, Y)$ vanishes on $S\Omega$.

iii) For any $\mathbf{f} = (f^1, \dots, f^m)^T \in C_c^\infty(\Omega)$, the function \mathbf{u} given by

$$\mathbf{u}(X) := \int_{\Omega} \tilde{\mathbf{G}}(Y, X) \mathbf{f}(Y) dY$$

belongs to $\dot{V}_2^{0,1}(\Omega)$ and satisfies $\mathcal{L}\mathbf{u} = \mathbf{f}$ in the sense of (2.4).

We remark that part iii) of the above definitions combined with the uniqueness of weak solutions of ${}^t\mathcal{L}\mathbf{u} = \mathbf{f}$ and $\mathcal{L}\mathbf{u} = \mathbf{f}$ in $\dot{V}_2^{0,1}(\Omega)$ for any $\mathbf{f} \in C_c^\infty(\Omega)$ gives uniqueness of Green's functions; see [4, §3.6] and [3].

3. MAIN RESULTS

The following condition (IH) means that weak solutions of $\mathcal{L}\mathbf{u} = 0$ and ${}^t\mathcal{L}\mathbf{u} = 0$ enjoy interior Hölder continuity estimates with an exponent μ_0 . It is not hard to see that this condition is equivalent to saying that the operator \mathcal{L} and its adjoint ${}^t\mathcal{L}$ satisfy the property (PH) in [4]; see [5, Lemma 8.2] for the proof.

Condition (IH). There exist constants $\mu_0 \in (0, 1]$, $R_c \in (0, \infty]$, and $C_0 > 0$ such that for all $X \in \Omega$ the following holds:

i) If \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $Q_-(X, R)$, where $R < R_c \wedge d^-(X)$, then we have

$$[\mathbf{u}]_{\mu_0/2, \mu_0; Q_-(X, R/2)} \leq C_0 R^{-\mu_0} \left(\int_{Q_-(X, R)} |\mathbf{u}|^2 \right)^{1/2}.$$

ii) If \mathbf{u} is a weak solution of ${}^t\mathcal{L}\mathbf{u} = 0$ in $Q_+(X, R)$, where $R < R_c \wedge d^+(X)$, then we have

$$[\mathbf{u}]_{\mu_0/2, \mu_0; Q_+(X, R/2)} \leq C_0 R^{-\mu_0} \left(\int_{Q_+(X, R)} |\mathbf{u}|^2 \right)^{1/2}.$$

By assuming the condition (IH), we construct the Green's function of \mathcal{L} in time-varying H_1 domains and the domains above time-varying H_1 graph in \mathbb{R}^{d+1} .

Theorem 3.1. *Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . Assume the condition (IH). Then there exists a unique Green's function $\mathbf{G}(X, Y) = \mathbf{G}(t, x, s, y)$ of \mathcal{L} in Ω . We have $\mathbf{G}(\cdot, Y) \in C_{loc}^{\mu_0/2, \mu_0}(\Omega \setminus \{Y\})$ for all $Y \in \Omega$ and*

$$(3.2) \quad \mathbf{G}(\cdot, Y) \equiv 0 \quad \text{on } \Omega(-\infty, s).$$

Also, there exists a unique Green's function $\tilde{\mathbf{G}}(X, Y)$ of ${}^t\mathcal{L}$ in Ω , which satisfies

$$(3.3) \quad \tilde{\mathbf{G}}(\cdot, Y) \equiv 0 \quad \text{on } \Omega(s, \infty)$$

and $\tilde{\mathbf{G}}(\cdot, Y) \in C_{loc}^{\mu_0/2, \mu_0}(\Omega \setminus \{Y\})$ for all $Y \in \Omega$. In addition, we have the following identity

$$(3.4) \quad \tilde{\mathbf{G}}(X, Y) := \mathbf{G}(Y, X)^T, \quad \forall X, Y \in \Omega, \quad X \neq Y.$$

Moreover, for any $\psi_0 \in L_2(\omega(s_0))^m$, the function $\mathbf{u}(t, x)$ given by

$$(3.5) \quad \mathbf{u}(t, x) = \int_{\omega(s_0)} \mathbf{G}(t, x, s_0, y) \psi_0(y) dy, \quad \forall X = (t, x) \in \Omega(s_0, \infty),$$

is a unique weak solution of the problem

$$(3.6) \quad \mathcal{L}\mathbf{u} = 0 \quad \text{in } \Omega(s_0, \infty), \quad \mathbf{u} = 0 \quad \text{on } S\Omega(s_0, \infty), \quad \mathbf{u} = \psi_0 \quad \text{on } \omega(s_0)$$

and if ψ_0 is continuous at $x_0 \in \omega(s_0)$ in addition, then

$$(3.7) \quad \lim_{\substack{(t, x) \rightarrow (s_0, x_0) \\ X \in \Omega(s_0, \infty)}} \int_{\omega(s_0)} \mathbf{G}(t, x, s_0, y) \psi_0(y) dy = \psi_0(x_0).$$

Furthermore, the following estimates hold for \mathbf{G} , where we denote $d'_Y = d(Y) \wedge R_c$:

- i) $\|\mathbf{G}(\cdot, Y)\|_{L_{2+4/d}(\Omega \setminus \overline{Q}(Y, R))} + \|\mathbf{G}(\cdot, Y)\|_{V_2(\Omega \setminus \overline{Q}(Y, R))} \leq NR^{-d/2}$ for all $R < d'_Y$ and $Y \in \Omega$.
- ii) $\|\mathbf{G}(\cdot, Y)\|_{L_p(\Omega[Y, R])} \leq NR^{-d+(d+2)/p}$ for all $r < d'_Y$, $Y \in \Omega$, and $p \in [1, \frac{d+2}{d})$.
- iii) $|\{X \in \Omega: |\mathbf{G}(X, Y)| > \tau\}| \leq N\tau^{-(d+2)/d}$ for all $\tau > (d'_Y/2)^{-d}$ and $Y \in \Omega$.
- iv) $\|D\mathbf{G}(\cdot, Y)\|_{L_p(\Omega[Y, R])} \leq NR^{-d-1+(d+2)/p}$ for all $r < d'_Y$, $Y \in \Omega$, and $p \in [1, \frac{d+2}{d+1})$.
- v) $|\{X \in \Omega: |D_x \mathbf{G}(X, Y)| > \tau\}| \leq N\tau^{-(d+2)/(d+1)}$ for all $\tau > (d'_Y/2)^{-d}$ and $Y \in \Omega$.
- vi) $|\mathbf{G}(X, Y)| \leq C|X - Y|_{\mathcal{D}}^{-d}$ whenever $0 < |X - Y|_{\mathcal{D}} < d'_Y/2$ and $X, Y \in \Omega$.
- vii) $|\mathbf{G}(X, Y) - \mathbf{G}(X', Y)| \leq C|X - X'|_{\mathcal{D}}^{\mu_0}|X - Y|_{\mathcal{D}}^{-d-\mu_0}$ whenever $2|X - X'|_{\mathcal{D}} < |X - Y|_{\mathcal{D}} < d'_Y/2$ and $X, X', Y \in \Omega$.

In the above, $N = N(d, m, v, \mu_0, C_0)$ and N depends on p as well in ii) and iv). The estimates i) – vii) are also valid for the Green's function $\tilde{\mathbf{G}}$ of the adjoint operator ${}^t\mathcal{L}$ in Ω .

Remark 3.8. In the condition (IH), the constant R_c is interchangeable with aR_c for any fixed $a \in (0, \infty)$, possibly at the cost of increasing the constant C_0 . Also, the condition (IH)

implies that if \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $Q_-(X_0, R)$ with $R < d^-(Y) \wedge R_c$, then we have the L_∞ estimate

$$(3.9) \quad \|\mathbf{u}\|_{L_\infty(Q_-(X_0, R/4))} \leq N \left(\int_{Q_-(X_0, R)} |\mathbf{u}|^2 \right)^{1/2},$$

where $N = N(d, m, \nu, \mu_0, C_0) > 0$. Moreover, \mathbf{u} satisfy

$$\|\mathbf{u}\|_{L_\infty(Q_-(X_0, r))} \leq N(R - r)^{-(d+2)/p} \|\mathbf{u}\|_{L_p(Q_-(X_0, R))}, \quad \forall r < R, \quad \forall p > 0,$$

where $N = N(d, m, \nu, \mu_0, C_0, p) > 0$. See [4, Lemma 2.6] for the proof.

Remark 3.10. In Theorem 3.1, we also have the following estimates, which follow from the identity (3.4) and the estimates *i) – vi)* for $\tilde{\mathbf{G}}(\cdot, X)$:

- i) $\|\mathbf{G}(X, \cdot)\|_{L_{2+4/d}(\Omega \setminus \bar{Q}(X, R))} + \|\mathbf{G}(X, \cdot)\|_{V_2(\Omega \setminus \bar{Q}(X, R))} \leq NR^{-d/2}$ for all $R < d'_X$ and $X \in \Omega$.
- ii) $\|\mathbf{G}(X, \cdot)\|_{L_p(\Omega[X, R])} \leq NR^{-d+(d+2)/p}$ for all $r < d'_X$, $X \in \Omega$, and $p \in [1, \frac{d+2}{d}]$.
- iii) $|\{Y \in \Omega : |\mathbf{G}(X, Y)| > \tau\}| \leq N\tau^{-(d+2)/d}$ for all $\tau > (d'_X/2)^{-d}$ and $X \in \Omega$.
- iv) $\|D\mathbf{G}(X, \cdot)\|_{L_p(\Omega[X, R])} \leq NR^{-d-1+(d+2)/p}$ for all $r < d'_X$, $X \in \Omega$, and $p \in [1, \frac{d+2}{d+1}]$.
- v) $|\{Y \in \Omega : |D_Y \mathbf{G}(X, Y)| > \tau\}| \leq N\tau^{-(d+2)/(d+1)}$ for all $\tau > (d'_X/2)^{-d}$ and $X \in \Omega$.
- vi) $|\mathbf{G}(X, Y)| \leq C|X - Y|_{\mathcal{D}}^{-d}$ whenever $0 < |X - Y|_{\mathcal{D}} < d'_X/2$ and $X, Y \in \Omega$.
- vii) $|\mathbf{G}(X, Y) - \mathbf{G}(X, Y')| \leq C|Y - Y'|_{\mathcal{D}}^{\mu_0} |X - Y|_{\mathcal{D}}^{-d-\mu_0}$ whenever $2|Y - Y'|_{\mathcal{D}} < |X - Y|_{\mathcal{D}} < d'_X/2$ and $X, Y, Y' \in \Omega$.

In particular, $|\mathbf{G}(X, Y)| \leq N|X - Y|_{\mathcal{D}}^{-d}$ whenever $0 < |X - Y|_{\mathcal{D}} < \frac{1}{2}(d(X) \vee d(Y)) \wedge R_c$.

The following condition (LB) is used to obtain a global Gaussian bound for the Green's function $\mathbf{G}(X, Y)$ in a time-varying H_1 domain $\Omega \subset \mathbb{R}^{d+1}$.

Condition (LB). There exist constants $R_{\max} \in (0, \infty]$ and $N_0 > 0$ so that for all $X \in \Omega$ and $0 < R < R_{\max}$, the following holds.

- i) If \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $\Omega_-[X, R]$ vanishing on $\mathcal{P}\Omega_-[X, R]$, then we have

$$\|\mathbf{u}\|_{L_\infty(\Omega_-[X, R/2])} \leq N_0 R^{-(2+d)/2} \|\mathbf{u}\|_{L_2(\Omega_-[X, R])}.$$

- ii) If \mathbf{u} is a weak solution of ${}^t\mathcal{L}\mathbf{u} = 0$ in $\Omega_+[X, R]$ vanishing on $\widetilde{\mathcal{P}}\Omega_+[X, R]$, then we have

$$\|\mathbf{u}\|_{L_\infty(\Omega_+[X, R/2])} \leq N_0 R^{-(2+d)/2} \|\mathbf{u}\|_{L_2(\Omega_+[X, R])}.$$

Theorem 3.11. Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . Assume the condition (LB) as well as the condition (IH). Then the Green's function $\mathbf{G}(X, Y)$ of \mathcal{L} in Ω exists and satisfies the conclusions of Theorem 3.1. Moreover, for all $X = (t, x)$ and $Y = (s, y)$ in Ω with $t > s$, we have

$$(3.12) \quad |\mathbf{G}(t, x, s, y)| \leq N \left\{ (t - s) \wedge R_{\max}^2 \right\}^{-d/2} \exp \left\{ -\kappa |x - y|^2 / (t - s) \right\},$$

where $N = N(d, m, \nu, N_0)$ and $\kappa = \kappa(\nu) > 0$.

Remark 3.13. In the condition (LB), the constant R_{\max} is interchangeable with aR_{\max} for any fixed $a \in (0, \infty)$, possibly at the cost of increasing the constant N_0 . In Theorem 3.11, the estimate (3.12) implies, via straightforward computation, that

$$(3.14) \quad |\mathbf{G}(X, Y)| \leq N|X - Y|_{\mathcal{D}}^{-d}, \quad \text{if } 0 < |t - s| < R_{\max}^2,$$

where $N = N(d, m, \nu, N_0)$. Then, similar to Lemma 5.8 below, one can show

$$(3.15) \quad \|\mathbf{G}(\cdot, Y)\|_{L_{2+4/d}(\Omega \setminus \bar{Q}(Y, R))} + \|\mathbf{G}(\cdot, Y)\|_{V_2(\Omega \setminus \bar{Q}(Y, R))} \leq NR^{-d/2}, \quad \forall R \in (0, R_{\max}),$$

where $N = N(d, m, \nu, N_0)$. Moreover, using (3.15) and proceeding as in [4, Section 4.2], one can show that \mathbf{G} satisfies the estimates $ii) - vi)$ in Theorem 3.1 with d'_Y replaced by R_{max} . Also, it is clear that the estimate (1.2) in the introduction follows from Theorem 3.11.

In order to derive the estimate (1.3) in the introduction, we introduce the following condition (LH) which, loosely speaking, says that weak solutions of $\mathcal{L}u = 0$ and ${}^t\mathcal{L}u = 0$ vanishing on $\Sigma \subset \partial\Omega$ are locally Hölder continuous up to Σ with exponent μ_0 .

Condition (LH). There exist $\mu_0 \in (0, 1]$, $R_{max} \in (0, \infty]$, and $N_1 > 0$ so that for all $X \in \Omega$ and $0 < R < R_{max}$, the following holds.

i) If u is a weak solution of $\mathcal{L}u = 0$ in $\Omega_-[X, R]$ vanishing on $\mathcal{P}\Omega_-[X, R]$, then we have

$$[\tilde{u}]_{\mu_0/2, \mu_0; Q_-(X, R/2)} \leq N_1 R^{-\mu_0} \left(\int_{Q_-(X, R)} |\tilde{u}|^2 \right)^{1/2}, \text{ where } \tilde{u} = \chi_{\Omega_-[X, R]} u.$$

ii) If u is a weak solution of ${}^t\mathcal{L}u = 0$ in $\Omega_+[X, R]$ vanishing on $\widetilde{\mathcal{P}}\Omega_+[X, R]$, then we have

$$[\tilde{u}]_{\mu_0/2, \mu_0; Q_+(X, R/2)} \leq N_1 R^{-\mu_0} \left(\int_{Q_+(X, R)} |\tilde{u}|^2 \right)^{1/2}, \text{ where } \tilde{u} = \chi_{\Omega_+[X, R]} u.$$

It is easy to see that the condition (LH) implies the condition (LB); see Lemma 6.1 in Appendix for the proof. Also, it is obvious that the condition (LH) implies the condition (IH). Therefore if the condition (LH) is satisfied, then there exists the Green's function of \mathcal{L} and it satisfies the conclusions of Theorems 3.1 and 3.11. The following theorem says that in fact, in such a case, a better estimate for the Green's function is available.

Theorem 3.16. Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . Assume the condition (LH). Then the Green's function $\mathbf{G}(X, Y)$ of \mathcal{L} in Ω exists and satisfies the conclusions of Theorem 3.1. Moreover, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have

$$(3.17) \quad |\mathbf{G}(t, x, s, y)| \leq N \delta(X, Y)^{\mu_0} \{(t - s) \wedge R_{max}^2\}^{-d/2} \exp\left\{-\kappa|x - y|^2/(t - s)\right\},$$

where $N = N(d, m, \nu, \mu_0, N_1)$ and $\kappa = \kappa(\nu) > 0$ and we used the notation

$$(3.18) \quad \delta(X, Y) = \left(1 \wedge \frac{d^-(X)}{R_{max} \wedge |X - Y|_{\mathcal{D}}}\right) \left(1 \wedge \frac{d^+(Y)}{R_{max} \wedge |X - Y|_{\mathcal{D}}}\right).$$

Remark 3.19. In the condition (LH), the constant R_{max} is interchangeable with aR_{max} for any fixed $a \in (0, \infty)$, possibly at the cost of increasing the constant N_1 . Also, we note that the estimate (1.3) in the introduction follows from Theorem 3.16 if Ω be a time-varying H_1 domain or a time-varying H_1 graph domain with $R_{max} = \infty$.

Remark 3.20. In Theorem 3.16, we also have the estimate

$$|\mathbf{G}(X, Y) - \mathbf{G}(X', Y)| \leq \frac{N \delta(X, Y)^{\mu_0}}{\{(t - s) \wedge R_{max}^2\}^{d/2}} \left(\frac{|X - X'|_{\mathcal{D}}}{|X - Y|_{\mathcal{D}}} \right)^{\mu_0} \exp\left\{-\frac{\kappa|x - y|^2}{t - s}\right\}$$

whenever $2|X - X'|_{\mathcal{D}} < |X - Y|_{\mathcal{D}}$ and $t > s$. It follows from (3.17) and the condition (LH).

4. SOME APPLICATIONS OF MAIN RESULTS

4.1. Scalar case. In the scalar case (i.e., $m = 1$), both conditions (LB) and (IH) are satisfied with $R_c = R_{max} = \infty$ and $N_0 = N_0(d, \nu)$; see e.g., [22, Chapter VI]. Also, in the scalar case, the Green's function is a nonnegative scalar function. Therefore, the following corollary is an immediate consequence of Theorem 3.11.

Corollary 4.1. *Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . If $m = 1$, then the Green's function $G(X, Y)$ of \mathcal{L} in Ω exists and satisfies the conclusions of Theorem 3.1 with d'_Y replaced by R_a . Moreover, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have*

$$G(t, x, s, y) \leq N(t-s)^{-d/2} \exp\{-\kappa|x-y|^2/(t-s)\},$$

where $N = N(d, \nu)$ and $\kappa = \kappa(\nu)$ are universal constants independent of Ω .

In fact, in the scalar case, a better estimate is available near the boundary. Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . By using the results in [22, §VI.8], one can show that in the case when $m = 1$, the condition (LH) is satisfied in Ω . Moreover, in the case when Ω is a time-varying H_1 graph domain, then the condition (LH) is satisfied with $R_{\max} = \infty$. Also, in that case, there exists $N = N(M) \geq 1$ such that

$$(4.2) \quad 1 \leq d^-(X)/d(X), \quad d^+(X)/d(X) \leq N, \quad \forall X \in \Omega.$$

Therefore, the following corollaries are immediate consequences of Theorem 3.16.

Corollary 4.3. *Assume that $m = 1$ and let $G(X, Y)$ be the Green's function of \mathcal{L} in Ω , where Ω is a time-varying H_1 domain in \mathbb{R}^{d+1} . Let $\delta(X, Y)$ be as defined in (3.18) with $R_{\max} = R_a$. Then, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have*

$$G(t, x, s, y) \leq N\delta(X, Y)^{\mu_0} \{(t-s) \wedge R_a^2\}^{-d/2} \exp\{-\kappa|x-y|^2/(t-s)\},$$

where $N = N(d, \nu)$ and $\kappa = \kappa(\nu)$ are positive constants independent of Ω .

Corollary 4.4. *Assume that $m = 1$ and let $G(X, Y)$ be the Green's function of \mathcal{L} in Ω , where Ω is a time-varying H_1 graph domain in \mathbb{R}^{d+1} . Then, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have*

$$G(t, x, s, y) \leq N \left(1 \wedge \frac{d(X)}{|X-Y|_{\mathcal{D}}}\right)^{\mu_0} \left(1 \wedge \frac{d(Y)}{|X-Y|_{\mathcal{D}}}\right)^{\mu_0} \frac{1}{(t-s)^{d/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\},$$

where $N = N(d, \nu, M)$ and $\kappa = \kappa(\nu)$ are positive constants.

4.2. L_∞ -perturbation of diagonal systems. Let $a^{\alpha\beta}(X)$ be scalar functions satisfying

$$(4.5) \quad a^{\alpha\beta}(X)\xi_\beta\xi_\alpha \geq \nu_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^d; \quad \sum_{\alpha,\beta=1}^d |a^{\alpha\beta}(X)|^2 \leq \nu_0^{-2},$$

for all $X \in \mathbb{R}^{d+1}$ with some constant $\nu_0 \in (0, 1]$. Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . Let $A_{ij}^{\alpha\beta}$ be the coefficients of the operator \mathcal{L} . We denote

$$(4.6) \quad \mathcal{E} = \sup_{X \in \mathbb{R}^{d+1}} \left\{ \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^d |A_{ij}^{\alpha\beta}(X) - a^{\alpha\beta}(X)\delta_{ij}|^2 \right\}^{1/2},$$

where δ_{ij} is the Kronecker delta symbol. By Lemma 6.8, there exists $\mathcal{E}_0 = \mathcal{E}_0(d, \nu_0, M)$ such that if $\mathcal{E} < \mathcal{E}_0$, then the condition (LH) is satisfied with $\mu_0 = \mu_0(d, \nu_0, M)$, $R_{\max} = R_a$, and $N_1 = N_1(d, m, \nu_0, M)$. Therefore, the following corollaries are another easy consequences of Theorem 3.16.

Corollary 4.7. *Let Ω be a time-varying H_1 domain in \mathbb{R}^{d+1} and let $\delta(X, Y)$ be as in (3.18) with $R_{\max} = R_a$. There exists $\mathcal{E}_0 = \mathcal{E}_0(d, \nu_0, M)$ such that if $\mathcal{E} < \mathcal{E}_0$, then the Green's function $G(X, Y)$ of \mathcal{L} in Ω exists and satisfies the conclusions of Theorem 3.1 with d'_Y replaced by R_a . Moreover, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have*

$$|G(t, x, s, y)| \leq N\delta(X, Y)^{\mu_0} \{(t-s) \wedge R_a^2\}^{-d/2} \exp\{-\kappa|x-y|^2/(t-s)\},$$

where N, μ_0 , and κ are constants depending on d, m, ν_0 , and M .

Corollary 4.8. *Let Ω be a time-varying H_1 graph domain in \mathbb{R}^{d+1} . There exists $\mathcal{E}_0 = \mathcal{E}_0(d, \nu_0, M)$ such that if $\mathcal{E} < \mathcal{E}_0$, then the Green's function $G(X, Y)$ of \mathcal{L} in Ω exists and satisfies the conclusions of Theorem 3.1 with d'_Y replaced by ∞ . Moreover, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have*

$$|G(t, x, s, y)| \leq N \left(1 \wedge \frac{d(X)}{|X - Y|_{\mathcal{D}}}\right)^{\mu_0} \left(1 \wedge \frac{d(Y)}{|X - Y|_{\mathcal{D}}}\right)^{\mu_0} \frac{1}{(t - s)^{d/2}} \exp\left\{-\frac{\kappa|x - y|^2}{t - s}\right\},$$

where N, μ_0 , and κ are constants depending on d, m, ν_0 , and M .

4.3. Systems with VMO_x coefficients. For a measurable function $f = f(X) = f(t, x)$ defined on \mathbb{R}^{d+1} , we set for $\rho > 0$

$$\omega_\rho(f) := \sup_{X \in \mathbb{R}^{d+1}} \sup_{r \leq \rho} \int_{t-r^2}^{t+r^2} \int_{B(x,r)} |f(y, s) - \bar{f}_{x,r}(s)| dy ds; \quad \bar{f}_{x,r}(s) = \int_{B(x,r)} f(s, \cdot).$$

We say that f belongs to VMO_x if $\lim_{\rho \rightarrow 0} \omega_\rho(f) = 0$. Note that VMO_x is a strictly larger class than the classical VMO space. In particular, VMO_x contains all functions uniformly continuous in x and measurable in t ; see [19].

By [4, Lemma 2.3], we find that if the coefficients of \mathcal{L} belong to VMO_x , then the condition (IH) is satisfied with parameters μ_0, N_0 , and R_c depending on $\omega_\rho(A^{\alpha\beta})$ as well as on d, m, ν . Therefore, we have the following corollary of Theorem 3.1.

Corollary 4.9. *Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . If the coefficients of \mathcal{L} belong to VMO_x , then the Green's function of \mathcal{L} exists in Ω and satisfies the conclusions of Theorem 3.1 with some $R_c > 0$.*

Remark 4.10. In Corollary 4.9, instead of assuming $A^{\alpha\beta} \in \text{VMO}_x$, one may assume that $\omega_\rho(A^{\alpha\beta})$ is sufficiently small for some $\rho > 0$. Also, if Ω is a time-varying domain satisfying the hypothesis of Section 2.3 with $f = f(X') = f(t, x') \in H_{1+\alpha}(\mathbb{R}^d)$ for some $\alpha > 0$, then one can show that the condition (LH) is satisfied with the parameters μ_0, N_1 , and $R_{\max} < \infty$ depending on d, m, ν , and $\omega_\rho(A^{\alpha\beta})$; see [22] for the definition of the space $H_{1+\alpha}$. Therefore, in that case, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$, we have

$$|G(t, x, s, y)| \leq N \delta(X, Y)^{\mu_0} \{(t - s) \wedge R_{\max}^2\}^{-d/2} \exp\{-\kappa|x - y|^2/(t - s)\},$$

where $\delta(X, Y)$ is as in (3.18).

5. PROOFS OF MAIN THEOREMS

5.1. Proof of Theorem 3.1. By following [4], we shall first construct the ‘‘averaged’’ Green's function of \mathcal{L} in Ω . Notice that we have $\partial\Omega = \mathcal{P}\Omega = \widetilde{\mathcal{P}}\Omega = S\Omega$. The following lemma is used for the construction of the averaged Green's function, which follows essentially from Brown et al. [3] and an embedding theorem in [20, §II.3], which is also valid for functions in $\dot{V}_2^{0,1}(\Omega(t_0, t_1))$. We remark that the function space $\dot{V}_2^{0,1}(\Omega)$ coincides with the function space $V_0(\Omega)$ used in [3].

Lemma 5.1. *For $\mathbf{g} \in C_c^\infty(\Omega(t_0, t_1))^m$ and $\psi_0 \in L_2(\omega(t_0))$, there exists a unique weak solution $\mathbf{v} \in \dot{V}_2^{0,1}(\Omega(t_0, t_1))$ of the problem*

$$\mathcal{L}\mathbf{v} = \mathbf{g} \text{ in } \Omega(t_0, t_1), \quad \mathbf{v} = 0 \text{ on } S\Omega(t_0, t_1), \quad \mathbf{v} = \psi_0 \text{ on } \omega(t_0).$$

Moreover, we have the following energy inequality for the weak solution \mathbf{v} :

$$(5.2) \quad \|\mathbf{v}\|_{V_2(\Omega(t_0, t_1))} \leq N \left(\|\mathbf{g}\|_{L_{(2d+4)/(d+4)}(\Omega(t_0, t_1))} + \|\psi_0\|_{L_2(\omega(t_0))} \right); \quad N = N(d, m, \nu).$$

Similarly, for $\mathbf{f} \in C_c^\infty(\Omega(t_0, t_1))^m$ and $\boldsymbol{\psi}_1 \in L_2(\omega(t_1))$, there exists a unique weak solution $\mathbf{u} \in \dot{V}_2^{0,1}(\Omega(t_0, t_1))$ of the problem

$${}^t\mathcal{L}\mathbf{u} = \mathbf{f} \text{ in } \Omega(t_0, t_1), \quad \mathbf{u} = 0 \text{ on } S\Omega(t_0, t_1), \quad \mathbf{u} = \boldsymbol{\psi}_1 \text{ on } \omega(t_1)$$

and \mathbf{u} satisfies the following energy inequality:

$$\|\mathbf{u}\|_{V_2(\Omega(t_0, t_1))} \leq N \left(\|\mathbf{f}\|_{L_{(2d+4)/(d+4)}(\Omega(t_0, t_1))} + \|\boldsymbol{\psi}_1\|_{L_2(\omega(t_1))} \right); \quad N = N(d, m, \nu).$$

Furthermore, we have the identity

$$(5.3) \quad \int_{\Omega(t_0, t_1)} \mathbf{f} \cdot \mathbf{v} \, dX + \int_{\omega(t_1)} \mathbf{v} \cdot \boldsymbol{\psi}_1 \, dx = \int_{\Omega(t_0, t_1)} \mathbf{u} \cdot \mathbf{g} \, dX + \int_{\omega(t_0)} \mathbf{u} \cdot \boldsymbol{\psi}_0 \, dx.$$

Let us fix a function $\Phi \in C_c^\infty(\mathbb{R}^{d+1})$ such that Φ is supported in $Q_-(0, 1)$, $0 \leq \Phi \leq 2$, and $\int_{\mathbb{R}^{d+1}} \Phi = 1$. Let $Y = (s, y) \in \Omega$ be fixed but arbitrary. For $0 < \varepsilon < d(Y)$, we define

$$\Phi_\varepsilon(X) = \Phi_\varepsilon(t, x) = \varepsilon^{-d-2} \Phi((t-s)/\varepsilon^2, (x-y)/\varepsilon).$$

Fix $t_0 \in (-\infty, s - \varepsilon^2)$ and let $\mathbf{v} = \mathbf{v}_{\varepsilon, Y, k}$ be a unique weak solution of the problem

$$\mathcal{L}\mathbf{v} = \Phi_\varepsilon \mathbf{e}_k \text{ in } \Omega(t_0, \infty), \quad \mathbf{v} = 0 \text{ on } S\Omega(t_0, \infty), \quad \mathbf{v} = 0 \text{ on } \omega(t_0),$$

where \mathbf{e}_k is the k -th unit vector in \mathbb{R}^m . By the uniqueness, we find that \mathbf{v} does not depend on the particular choice of t_0 and we may extend \mathbf{v} to the entire Ω by setting

$$(5.4) \quad \mathbf{v} = \mathbf{v}_{\varepsilon, Y, k} \equiv 0 \quad \text{on} \quad \Omega(-\infty, s - \varepsilon^2).$$

Then $\mathbf{v} \in \dot{V}_2^{0,1}(\Omega)$ and satisfies for all $\tau > s$ the identity

$$\int_{\omega(\tau)} v^i \phi^i \, dx - \int_{\Omega(-\infty, \tau)} v^i \phi_t^i \, dX + \int_{\Omega(-\infty, \tau)} A_{ij}^{\alpha\beta} D_\beta v^j D_\alpha \phi^i \, dX = \int_{Q_-(Y, \varepsilon)} \Phi_\varepsilon \phi^k \, dX$$

for all $\phi \in C_c^\infty(\Omega)^m$. We define the averaged Green's function $\mathbf{G}^\varepsilon(\cdot, Y) = (G_{jk}^\varepsilon(\cdot, Y))_{j,k=1}^m$ of the operator \mathcal{L} in Ω by

$$\mathbf{G}_{jk}^\varepsilon(\cdot, Y) = v^j = v_{\varepsilon, Y, k}^j.$$

Notice that by Lemma 5.1 and an embedding theorem (see [20, §II.3]) we obtain

$$(5.5) \quad \|\mathbf{G}^\varepsilon(\cdot, Y)\|_{L_{2+4/d}(\Omega)} \leq N \|\mathbf{G}^\varepsilon(\cdot, Y)\|_{V_2(\Omega)} \leq N \|\Phi_\varepsilon\|_{L_{(2d+4)/(d+4)}(\Omega)} \leq N \varepsilon^{-d(d+2)/(2d+4)}.$$

Next, for any given $\mathbf{f} \in C_c^\infty(\Omega)^m$, fix t_1 such that $\mathbf{f} \equiv 0$ on $\Omega(t_1, \infty)$ and let \mathbf{u} be a unique weak solution of the problem

$${}^t\mathcal{L}\mathbf{u} = \mathbf{f} \text{ in } \Omega(-\infty, t_1), \quad \mathbf{u} = 0 \text{ on } S\Omega(-\infty, t_1), \quad \mathbf{u} = 0 \text{ on } \omega(t_1),$$

Again, by the uniqueness we may extend \mathbf{u} to the entire Ω by setting $\mathbf{u} \equiv 0$ on $\Omega(t_1, \infty)$.

Then, $\mathbf{u} \in \dot{V}_2^{0,1}(\Omega)$ and satisfies for all τ the identity

$$\int_{\omega(\tau)} u^i \phi^i \, dx + \int_{\Omega(\tau, \infty)} u^i \phi_t^i \, dX + \int_{\Omega(\tau, \infty)} \tilde{A}_{ij}^{\alpha\beta} D_\beta v^j D_\alpha \phi^i \, dX = \int_{\Omega(\tau, \infty)} f^i \phi^i \, dX$$

for all $\phi \in C_c^\infty(\Omega)$. Also, similar to (5.5), we have

$$\|\mathbf{u}\|_{L_{2+4/d}(\Omega)} \leq N \|\mathbf{f}\|_{L_{(2d+4)/(d+4)}(\Omega)}.$$

Now, let $X_0 \in \Omega$ and $R < d(X_0) \wedge R_c$ be fixed but arbitrary, and assume that \mathbf{f} is supported in $Q_+(X_0, R) \subset \Omega$. By using the condition (IH) and following the same argument as in [4, Section 3.2], we obtain

$$(5.6) \quad \|\mathbf{u}\|_{L_\infty(Q_+(X_0, R/4))} \leq N R^{2-(d+2)/p} \|\mathbf{f}\|_{L_p(Q_+(X_0, R))}, \quad \forall p > (d+2)/2.$$

If $Q_-(Y, \varepsilon) \subset Q_+(X_0, R/4)$, then (5.3) together with (5.6) yields

$$\left| \int_{Q_+(X_0, R)} \mathbf{G}^\varepsilon(\cdot, Y) f \right| \leq \int_{Q_-(Y, \varepsilon)} \Phi_\varepsilon |u| \leq NR^{2-(d+2)/p} \|f\|_{L_p(Q_+(X_0, R))}, \quad \forall p > (d+2)/2.$$

By duality, it follows that if $Q_-(Y, \varepsilon) \subset Q_+(X_0, R/4)$, then we have

$$\|\mathbf{G}^\varepsilon(\cdot, Y)\|_{L_q(Q_+(X_0, R))} \leq NR^{-d+(d+2)/q}, \quad \forall q \in [1, (d+2)/d].$$

Then, by following the proof of [4, Lemma 3.2], we conclude

$$(5.7) \quad |\mathbf{G}^\varepsilon(X, Y)| \leq N|X - Y|_{\mathcal{D}}^{-d}, \quad \forall \varepsilon \leq \frac{1}{3}|X - Y|_{\mathcal{D}} \text{ if } |X - Y|_{\mathcal{D}} < \frac{1}{2}(d(Y) \wedge R_c).$$

The following lemma is a consequence of the energy inequality of Brown et al. [3], the above estimate (5.7), and an embedding theorem in [20, §II.3].

Lemma 5.8. *For $R < \frac{1}{2}(d(Y) \wedge R_c)$, let $\zeta \in C_c^\infty(Q(Y, R))$ be a cut-off function such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ on $Q(Y, R/2)$. Then, for all $\varepsilon > 0$ we have*

$$\|(1 - \zeta)\mathbf{G}^\varepsilon(\cdot, Y)\|_{V_2(\Omega)} \leq N(\|D\zeta\|_{L^\infty}^2 + \|\zeta_t\|_{L^\infty})^{1/2} R^{1-d/2}.$$

In particular, for all $\varepsilon > 0$ and $R < \frac{1}{2}(d(Y) \wedge R_c)$, we have

$$\|\mathbf{G}^\varepsilon(\cdot, Y)\|_{V_2(\Omega \setminus \overline{Q}(Y, R))} \leq NR^{-d/2}.$$

The following lemma is an analogue of [4, Lemma 6.1] in time-varying H^1 domains, the proof of which is essentially the same.

Lemma 5.9. *Let $\{u_k\}_{k=1}^\infty$ be a sequence in $V_2(\Omega)$. If $\sup_k \|u_k\|_{V_2(\Omega)} \leq N < \infty$, then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty \subseteq \{u_k\}_{k=1}^\infty$ and $u \in V_2(\Omega)$ with $\|u\|_{V_2(\Omega)} \leq N$ such that $u_{k_j} \rightharpoonup u$ weakly in $W_2^{0,1}(\Omega(t_0, t_1))$ for all $-\infty < t_0 < t_1 < \infty$. Moreover, if all u_k vanish on $S\Omega$, then u also vanishes on $S\Omega$.*

The above two lemmas contain all ingredients needed for the construction of a Green's function. By following the argument in [4, Section 4.2] verbatim, we construct the Green's function $\mathbf{G}(\cdot, Y)$ from $\mathbf{G}^\varepsilon(\cdot, Y)$, and it is readily seen that $\mathbf{G}(\cdot, Y) \in C_{loc}^{\mu_0/2, \mu_0}(\Omega \setminus \{Y\})$ satisfies (3.2) as well as the estimates *i) – vi)*. The estimate *vii)* does not appear explicitly in [4] but it easily follows from the estimates *vi)* and the condition (IH); see [14, §3.6].

Also, fix a function $\Psi \in C_c^\infty(\mathbb{R}^{d+1})$ such that Ψ is supported in $Q_+(0, 1)$, $0 \leq \Psi \leq 2$, and $\int_{\mathbb{R}^{d+1}} \Psi = 1$. For $0 < \varepsilon < d(Y)$, where $Y = (s, y) \in \Omega$ be fixed but arbitrary, we set

$$\Psi_\varepsilon(X) = \Psi_\varepsilon(t, x) = \varepsilon^{-d-2} \Psi((t-s)/\varepsilon^2, (x-y)/\varepsilon).$$

Fix $t_1 \in (t + \varepsilon^2, \infty)$ and let $\mathbf{w} = \mathbf{w}_{\varepsilon, Y, k}$ be a unique weak solution of the problem

$${}^t\mathcal{L}\mathbf{w} = \Psi_\varepsilon e_k \text{ in } \Omega(-\infty, t_1), \quad \mathbf{w} = 0 \text{ on } S\Omega(-\infty, t_1), \quad \mathbf{w} = 0 \text{ on } \omega(t_1),$$

Then, as before, we may extend \mathbf{w} to the entire Ω by setting $\mathbf{w} \equiv 0$ on $\Omega(t + \varepsilon^2, \infty)$ so that \mathbf{w} belongs to $\mathring{V}_2^{1,0}(\Omega)$ and satisfies for all $\tau < t$ the identity

$$(5.10) \quad \int_{\omega(\tau)} w^i \phi^i dx + \int_{\Omega(\tau, \infty)} w^i \phi_i^i dX + \int_{\Omega(\tau, \infty)} \tilde{A}_{ij}^{\alpha\beta} D_\beta w^j D_\alpha \phi^i dX = \int_{Q_+(X, \varepsilon)} \Psi_\varepsilon \phi^k dX$$

for all $\phi \in C_c^\infty(\Omega)^m$. We define the averaged Green's function $\tilde{\mathbf{G}}^\varepsilon(\cdot, Y) = (\tilde{G}_{jk}^\varepsilon(\cdot, Y))_{j,k=1}^m$ of the adjoint operator ${}^t\mathcal{L}$ in Ω by

$$\tilde{G}_{jk}^\varepsilon(\cdot, Y) = w^j = w_{\varepsilon, Y, k}^j.$$

Then by a similar argument, we construct a Green's function $\tilde{G}(\cdot, Y)$ from $\tilde{G}^\varepsilon(\cdot, Y)$, which belongs to $C_{loc}^{\mu_0/2, \mu_0}(\Omega \setminus \{Y\})$ and satisfies (3.3) and the estimates $i) - vi)$. Moreover, by following [4, Lemma 3.5], we obtain the identity (3.4).

Next, we shall prove the identity (3.5). Let $\psi_0 \in L_2(\omega(s_0))^m$ be given and let \mathbf{u} be a unique weak solution of the problem (3.6). Fix $X = (t, x) \in \Omega(s_0, \infty)$ and let $\mathbf{w} = \mathbf{w}_{\varepsilon, X, k}$ be as constructed above. By Lemma 5.1, for ε sufficiently small, we have

$$(5.11) \quad \int_{Q_+(X, \varepsilon)} \Psi_\varepsilon \mathbf{u}^k dY = \int_{\omega(s_0)} \mathbf{w}_{\varepsilon, X, k} \cdot \psi_0 dy.$$

For $\psi_0 \in C_c^\infty(\omega(s_0))$, it can be easily seen that (see [4, Section 3.5])

$$\lim_{\varepsilon \rightarrow 0} \int_{\omega(s_0)} \mathbf{w}_{\varepsilon, X, k} \cdot \psi_0 dy = \int_{\omega(s_0)} \tilde{G}(\cdot, X) \mathbf{e}_k \cdot \psi_0 dy,$$

Since the condition (IH) implies that \mathbf{u} is continuous at X , by taking the limit $\varepsilon \rightarrow 0$ in (5.11) and using (3.4), we obtain

$$\mathbf{u}^k(X) = \int_{\omega(s_0)} \tilde{G}_{ik}(s_0, y, t, x) \psi_0^i(y) dy = \int_{\omega(s_0)} G_{ki}(t, x, s_0, y) \psi_0^i(y) dy.$$

We have thus derived (3.5) under an assumption that $\psi_0 \in C_c^\infty(\omega(s_0))$. For $\psi_0 \in L_2(\omega(s_0))^m$, let $\{\psi_j\}_{j=1}^\infty$ be a sequence in $C_c^\infty(\omega(s_0))^m$ such that $\psi_j \rightarrow \psi_0$ in $L_2(\omega(s_0))$. Let \mathbf{u}_j be a unique weak solution of the problem (3.6) with ψ_0 replaced by ψ_j . Then by Lemma 5.1, we find that $\lim_{j \rightarrow \infty} \|\mathbf{u}_j - \mathbf{u}\|_{V_2(\Omega(s_0, t))} = 0$ and by the condition (IH) and (3.9) we have $\lim_{j \rightarrow \infty} |\mathbf{u}_j(X) - \mathbf{u}(X)| = 0$. On the other hand, by the estimate $i)$ applied to $\tilde{G}(\cdot, X)$ together with the identity (3.4), we find that $\|G(t, x, s_0, \cdot)\|_{L_2(\omega(s_0))} < \infty$, and thus we get

$$\lim_{j \rightarrow \infty} \int_{\omega(s_0)} G(t, x, s_0, y) \psi_j(y) dy = \int_{\omega(s_0)} G(t, x, s_0, y) \psi_0(y) dy.$$

This completes the proof of (3.5). Similarly, for $\psi_1 \in L_2(\omega(t_1))^m$, let \mathbf{u} be a unique weak solution of the problem

$$\mathcal{L}\mathbf{v} = 0 \text{ in } \Omega(-\infty, t_1), \quad \mathbf{v} = 0 \text{ on } S\Omega(-\infty, t_1), \quad \mathbf{v} = \psi_1 \text{ on } \omega(t_1).$$

Then as above, \mathbf{v} has the following representation:

$$\mathbf{v}(s, y) = \int_{\omega(t_1)} \tilde{G}(s, y, t_1, x) \psi_1(x) dx.$$

It only remains us to prove (3.7). We proceed similar to [4, Section 4.4]. The following lemma is another simple consequence of Brown et al. [3].

Lemma 5.12. *Let $\eta = \eta(x) \in C^1(\mathbb{R}^d)$ be a bounded nonnegative function. Assume that $\mathbf{u} \in \dot{V}_2^{0,1}(\Omega(s_0, \infty))$ is the weak solution of the problem (3.6) and define*

$$I(t) = \frac{1}{2} \int_{\omega(t)} \eta(x) |\mathbf{u}(t, x)|^2 dx, \quad t \in (s_0, \infty).$$

Then $I(t)$ is absolutely continuous and satisfies a.e. $t > s_0$ the identity

$$I'(t) = - \int_{\omega(t)} A_{ij}^{\alpha\beta} D_\beta \mathbf{u}^j D_\alpha (\eta \mathbf{u}^i).$$

The following lemmas are key ingredients to prove (3.7) and adapted from [4].

Lemma 5.13. Assume that $\psi_0 \in L_2(\omega(s_0))^m$ is supported in a closed set $F \subset \overline{\omega(s_0)}$ and let \mathbf{u} be the weak solution of the problem (3.6). Then, we have

$$(5.14) \quad \int_E |\mathbf{u}(t, x)|^2 dx \leq e^{-\gamma \text{dist}(E, F)^2 / (t-s_0)} \int_F |\psi_0(x)|^2 dx, \quad \forall E \subset \omega(t),$$

where $\text{dist}(E, F) = \inf\{|x - y| : x \in E, y \in F\}$ and $\gamma = \gamma(\nu) > 0$.

Proof. We may assume that $\text{dist}(E, F) > 0$; otherwise (5.14) is an immediate consequence of the energy inequality (5.2). Let $\phi = \phi(x)$ be a bounded C^1 function on \mathbb{R}^d satisfying $|D\phi| \leq K$ for some $K > 0$ to be fixed later. Define

$$I(t) = \int_{\omega(t)} e^{2\phi(x)} |\mathbf{u}(t, x)|^2 dx, \quad t > s_0.$$

By Lemma 5.12, we find that $I'(t)$ satisfies for a.e. $t > s_0$

$$\begin{aligned} I'(t) &= -2 \int_{\omega(t)} e^{2\phi} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha u^i dx - 4 \int_{\omega(t)} e^{2\phi} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi u^i dx \\ &\leq -2\nu \int_{\omega(t)} e^{2\phi} |D\mathbf{u}|^2 dx + 4(K/\nu) \int_{\omega(t)} e^\phi |D\mathbf{u}| e^\phi |\mathbf{u}| dx \\ &\leq (2/\nu^3) K^2 \int_{\omega(t)} e^{2\phi} |\mathbf{u}|^2 dx = (2/\nu^3) K^2 I(t). \end{aligned}$$

The above differential inequality yields

$$(5.15) \quad I(t) \leq e^{(2/\nu^3) K^2 (t-s_0)} \|e^\phi \psi_0\|_{L_2(F)}^2, \quad \forall t \geq s_0.$$

Notice that by a standard approximation, we may assume that ϕ is a bounded Lipschitz continuous function satisfying $|D\phi| \leq K$ a.e. Since F is a closed set, the function

$$\text{dist}(x, F) = \inf\{|x - y| : y \in F\}$$

is a Lipschitz function on \mathbb{R}^d with Lipschitz constant 1 and $\text{dist}(E, F) = \inf_{x \in E} \text{dist}(x, F)$. Therefore, if we set $\phi(x) = K(\text{dist}(x, F) \wedge \text{dist}(E, F))$, then by (5.15), we get

$$\int_E |\mathbf{u}(t, x)|^2 dx \leq \exp\left\{(2/\nu^3) K^2 (t - s_0) - 2K \text{dist}(E, F)\right\} \int_F |\psi_0(x)|^2 dx.$$

The lemma follows if we set $K = \text{dist}(E, F) / \{(2/\nu^3)(t - s_0)\}$. ■

Lemma 5.16. Let \mathbf{u} be the weak solution of the problem (3.6), where $\psi_0 \in L_\infty(\omega(s_0))$ and has a compact support in $\omega(s_0)$. Denote

$$(5.17) \quad \varrho = \varrho(x) = \text{dist}(x, \partial\omega(s_0)) \wedge R_c, \quad x \in \omega(s_0)$$

Then for all $x \in \omega(s_0)$, we have

$$|\mathbf{u}(t, x)| \leq N \|\psi_0\|_{L_\infty(\omega(s_0))} \text{ whenever } 0 < t - s_0 < (1 \wedge M^{-2}) \varrho^2(x)/4,$$

where $N = N(d, m, \nu, \mu_0, C_0) > 0$.

Proof. For $x \in \omega(s_0)$, set $r = \varrho(x)/2$ and $\delta = (r/M)^2$ so that

$$(s_0 - \delta, s_0 + \delta) \times B(x, r) \subset \subset \Omega.$$

For any t satisfying $0 < t - s_0 < \delta \wedge r^2 = (1 \wedge M^{-2}) \varrho^2(x)/4$, set $R = \sqrt{t - s_0}$ and denote

$$A_0 = B(x, R); \quad A_k = \{y \in \omega(s_0) : 2^{k-1}R \leq |y - x| < 2^k R\}, \quad k = 1, 2, \dots$$

Since ψ_0 is compactly supported in $\omega(s_0)$, we have $\psi_0 = \sum_{k=0}^{k_0} \chi_{A_k} \psi_0$ for some $k_0 < \infty$. For $k = 0, 1, \dots, k_0$, we define

$$\mathbf{u}_k(t, x) = \int_{A_k} \mathbf{G}(t, x, s_0, y) \psi_0(y) dy.$$

Then, it follows from (3.5) that $\mathbf{u} = \sum_{k=0}^{k_0} \mathbf{u}_k$ and that each \mathbf{u}_k is the weak solution of the problem (3.6) with $\chi_{A_k} \psi_0$ in place of ψ_0 . We apply Lemma 5.1 to \mathbf{u}_k with $E = B(x, R)$ and $F = \bar{A}_k$ for $k = 1, 2, \dots$, to obtain that

$$\int_{B(x, R)} |\mathbf{u}_k(s, y)|^2 dy \leq N e^{-\gamma(2^{k-1}-1)} 2^{kd} R^d \|\psi_0\|_{L^\infty(\omega(s_0))}^2, \quad \forall s \in (s_0, t).$$

Therefore, by the condition (IH) and (3.9), we get

$$|\mathbf{u}(t, x)| \leq \sum_{k=0}^{k_0} |\mathbf{u}_k(t, x)| \leq N \left(1 + \sum_{k=1}^{\infty} e^{-\gamma(2^{k-1}-1)} 2^{kd/2} \right) \|\psi_0\|_{L^\infty(\omega(s_0))} \leq N \|\psi_0\|_{L^\infty(\omega(s_0))}.$$

The lemma is proved. \blacksquare

Lemma 5.18. *Let $\eta \in C_c^\infty(B(x_0, 4r))$ be a function satisfying*

$$(5.19) \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B(x_0, 2r), \quad \text{and } |D\eta| \leq 4/r,$$

where $x_0 \in \omega(s_0)$ and $r < \varrho(x_0)/5$, where ϱ is as defined in (5.17). Then, we have

$$\lim_{t \rightarrow s_0} \int_{\omega(s_0)} \mathbf{G}(t, x, s_0, y) \eta(y) dy = I_m, \quad \forall x \in B(x_0, r).$$

Proof. By taking $\varepsilon \rightarrow 0$ in (5.10) and arguing as in the proof of [4, Lemma 4.3], we find that the following identity holds for all $\tau < t$:

$$(5.20) \quad \phi^k(X) = \phi^k(t, x) = \int_{\omega(\tau)} \tilde{G}_{ik}(\tau, y, t, x) \phi^i(\tau, y) dy + \int_{\Omega(\tau, t)} \tilde{G}_{ik}(Y, X) \phi_s^i(Y) dY \\ + \int_{\Omega(\tau, t)} \tilde{A}_{ij}^{\alpha\beta} D_{y^\beta} \tilde{G}_{jk}(Y, X) D_\alpha \phi^i(Y) dY, \quad \forall \phi \in C_c^\infty(\Omega)^m,$$

where we have used (3.3). Let $\zeta = \zeta(s)$ be a smooth function on \mathbb{R} such that

$$0 \leq \zeta \leq 1, \quad \zeta(s) = 1 \text{ for } |s - s_0| \leq \delta, \quad \text{and } \zeta(s) = 0 \text{ for } |s - s_0| \geq 2\delta,$$

where δ is chosen so small that

$$(s_0 - 2\delta, s_0 + 2\delta) \times B(x_0, 4r) \subset \subset \Omega.$$

Notice that we may take $\phi(Y) = \phi(s, y) = \zeta(s) \eta(y) \mathbf{e}_i$ in (5.20). Setting $\tau = s_0$ and assuming that $|t - s_0| < \delta$ in (5.20), we obtain by (3.4) that

$$(5.21) \quad \delta_{kl} = \int_{\omega(s_0)} G_{kl}(t, x, s_0, y) \eta(y) dy + \int_{\Omega(s_0, t)} \tilde{A}_{ij}^{\alpha\beta} D_{y^\beta} \tilde{G}_{jk}(Y, X) D_\alpha \eta(y) dY =: I + II.$$

Then for all $X = (t, x)$ such that $x \in B(x_0, r)$ and $|t - s_0| < \delta \wedge r^2$, we estimate II as follows by using the hypothesis (5.19), Hölder's inequality, and the estimate $i)$ for $\tilde{G}(\cdot, X)$:

$$|II| \leq N r^{d/2-1} (t - s_0)^{1/2} \left(\int_{\Omega(s_0, t) \setminus \bar{Q}(X, r)} |D_y \tilde{G}(Y, X)|^2 dY \right)^{1/2} \leq C r^{-1} (t - s_0)^{1/2}.$$

Therefore, the lemma follows by taking the limit t to s_0 in (5.21). \blacksquare

We are ready to prove (3.7). Let $\psi_0 \in L_2(\omega(s_0))^m$ and assume that ψ_0 is continuous at $x_0 \in \omega(s_0)$. Let \mathbf{u} be the weak solution of the problem (3.6). For any $\varepsilon > 0$ given, choose $r < \varrho(x_0)/5$, where ϱ is as defined in (5.17), such that

$$|\psi_0(x) - \psi_0(x_0)| < \varepsilon/2N \quad \text{for all } x \text{ satisfying } |x - x_0| < 4r,$$

where N is the constant that appears in Lemma 5.16. Let η be given as in Lemma 5.18 and let \mathbf{u}_0 , \mathbf{u}_ε , and \mathbf{u}_∞ , respectively, be the weak solution of the problem (3.6) with $\eta\psi_0(x_0)$, $\eta(\psi_0 - \psi_0(x_0))$, and $(1 - \eta)\psi_0$ in place of ψ_0 . By the uniqueness, we have $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\varepsilon + \mathbf{u}_\infty$ and by the formula (3.5), \mathbf{u}_0 is represented by

$$(5.22) \quad \mathbf{u}_0(t, x) = \left(\int_{\omega(s_0)} \mathbf{G}(t, x, s_0, y) \eta(y) dy \right) \psi_0(x_0).$$

Let $\delta > 0$ be chosen so that

$$(s_0 - \delta, s_0 + \delta) \times B(x_0, 4r) \subset \subset \Omega.$$

For any s satisfying $0 < s - s_0 < \delta$, we set $E = B(x_0, r) \subset \omega(s)$ and $F = \overline{\omega(s_0)} \setminus B(x_0, 2r)$ in Lemma 5.13 to get

$$\int_{B(x_0, r)} |\mathbf{u}_\infty(s, y)|^2 dy \leq e^{-\gamma r^2/(s-s_0)} \|\psi_0\|_{L_2(\omega(s_0))}^2.$$

Therefore, for all t satisfying $0 < t - s_0 < \delta \wedge r^2$, we set $R = \sqrt{t - s_0}/4$ in (3.9) to get

$$(5.23) \quad |\mathbf{u}_\infty(t, x)| \leq NR^{-d/2} e^{-\gamma(r/R)^2} \|\psi_0\|_{L_2(\omega(s_0))}, \quad \forall x \in B(x_0, R).$$

Finally, we estimate \mathbf{u}_ε by using Lemma 5.16.

$$(5.24) \quad |\mathbf{u}_\varepsilon(t, x)| \leq \varepsilon/2 \quad \text{whenever } 0 < t - s_0 < (1 \wedge M^{-2})\varrho^2(x)/4.$$

Combining (5.22), (5.23), and (5.24), we see that if $t - s_0$ is chosen sufficiently small, then there exists $\vartheta > 0$ such that for all $x \in B(x_0, \vartheta)$ we have $|\mathbf{u}(t, x) - \psi_0(x_0)| < \varepsilon$. This completes the proof. \blacksquare

5.2. Proof of Theorem 3.11. By Theorem 3.1, the condition (IH) implies existence of the Green's function $\mathbf{G}(X, Y)$ of \mathcal{L} in Ω . Therefore, all the conclusions of Theorem 3.1 are satisfied. Let ϕ be a bounded Lipschitz function on \mathbb{R}^d satisfying $|D\phi| \leq K$ a.e. for some $K > 0$ to be chosen later. For any $\mathbf{f} \in L_2(\omega(s))^m$, let \mathbf{u} be a unique weak solution of the problem

$$(5.25) \quad \mathcal{L}\mathbf{u} = 0 \quad \text{in } \Omega(s, \infty), \quad \mathbf{u} = 0 \quad \text{on } S\Omega(s, \infty), \quad \mathbf{u} = e^{-\phi}\mathbf{f} \quad \text{on } \omega(s).$$

For $t > s$, we define the operator $P_{s \rightarrow t}^\phi : L_2(\omega(s))^m \rightarrow L_2(\omega(t))^m$ by

$$P_{s \rightarrow t}^\phi \mathbf{f}(x) = e^{\phi(x)} \mathbf{u}(t, x).$$

Notice that by the representation formula (3.5), we have

$$(5.26) \quad P_{s \rightarrow t}^\phi \mathbf{f}(x) = e^{\phi(x)} \int_{\omega(s)} \mathbf{G}(t, x, s, y) e^{-\phi(y)} \mathbf{f}(y) dy.$$

For $t \geq s$, we define

$$I(t) = \|e^{\phi} \mathbf{u}(t, \cdot)\|_{L_2(\omega(t))}^2 = \|P_{s \rightarrow t}^\phi \mathbf{f}\|_{L_2(\omega(t))}^2.$$

Then, as in the proof of Lemma 5.13, we find that I is absolutely continuous and satisfies for a.e. $t > s$, the differential inequality

$$I'(t) \leq (2/\nu^3)K^2 I(t).$$

The above inequality with the initial condition $I(s) = \|f\|_{L_2(\omega(s))}^2$ yields

$$I(t) \leq e^{(2/\nu^3)K^2(t-s)} \|f\|_{L_2(\omega(s))}^2, \quad \forall t \geq s.$$

We have thus shown that for all $f \in L_2(\omega(s))^m$, the operator $P_{s \rightarrow t}^\phi$ satisfies

$$(5.27) \quad \|P_{s \rightarrow t}^\phi f\|_{L_2(\omega(t))} \leq e^{\vartheta K^2(t-s)} \|f\|_{L_2(\omega(s))}, \quad \forall t > s,$$

where $\vartheta = \nu^{-3}$. We set $R = \sqrt{t-s} \wedge R_{\max}$ and use the condition (LB) to estimate

$$\begin{aligned} e^{-2\phi(x)} |P_{s \rightarrow t}^\phi f(x)|^2 &= |u(t, x)|^2 = |u(X)|^2 \\ &\leq N_0^2 R^{-(d+2)} \int_{\Omega_{-[X, R]}} |u(Y)|^2 dY \\ &= N_0^2 R^{-(d+2)} \int_{t-R^2}^t \int_{\omega(\tau) \cap B(x, R)} e^{-2\phi(y)} |P_{s \rightarrow \tau}^\phi f(y)|^2 dy d\tau, \end{aligned}$$

Therefore, by using the estimate (5.27), we get

$$\begin{aligned} |P_{s \rightarrow t}^\phi f(x)|^2 &\leq N_0^2 R^{-d-2} \int_{t-R^2}^t \int_{\omega(\tau) \cap B(x, R)} e^{2\phi(x)-2\phi(y)} |P_{s \rightarrow \tau}^\phi f(y)|^2 dy d\tau \\ &\leq N_0^2 R^{-d-2} \int_{t-R^2}^t \int_{\omega(\tau) \cap B(x, R)} e^{2KR} |P_{s \rightarrow \tau}^\phi f(y)|^2 dy d\tau \\ &\leq N_0^2 R^{-d-2} e^{2KR} \int_{t-R^2}^t e^{2\vartheta K^2(\tau-s)} \|f\|_{L_2(\omega(s))}^2 d\tau \\ &\leq N_0^2 R^{-d} e^{2KR+2\vartheta K^2(t-s)} \|f\|_{L_2(\omega(s))}^2. \end{aligned}$$

We have thus obtained the following $L_2(\omega(s)) \rightarrow L_\infty(\omega(t))$ estimate for $P_{s \rightarrow t}^\phi$:

$$(5.28) \quad \|P_{s \rightarrow t}^\phi f\|_{L_\infty(\omega(t))} \leq N_0 R^{-d/2} e^{KR+\vartheta K^2(t-s)} \|f\|_{L_2(\omega(s))}.$$

We also define the operator $Q_{t \rightarrow s}^\phi : L_2(\omega(t))^m \rightarrow L_2(\omega(s))^m$ for $s < t$ by setting

$$Q_{t \rightarrow s}^\phi g(y) = e^{-\phi(y)} v(s, y), \quad \forall g \in L_2(\omega(t))^m,$$

where v is a unique weak solution of the problem

$$(5.29) \quad \begin{cases} \mathcal{L}v = 0 & \text{in } \Omega(-\infty, t), \\ v = 0 & \text{on } S\Omega(-\infty, t), \\ v = e^\phi g & \text{on } \omega(t). \end{cases}$$

By a similar calculation that leads to (5.28), we obtain

$$(5.30) \quad \|Q_{t \rightarrow s}^\phi g\|_{L_\infty(\omega(s))} \leq N_0 R^{-d/2} e^{KR+\vartheta K^2(t-s)} \|g\|_{L_2(\omega(t))}.$$

It follows from (5.25), (5.29), and (5.3) in Lemma 5.1 that

$$\int_{\omega(t)} (P_{s \rightarrow t}^\phi f) \cdot g dx = \int_{\omega(s)} f \cdot (Q_{t \rightarrow s}^\phi g) dx, \quad \forall f \in L_2(\omega(s))^m, \quad \forall g \in L_2(\omega(t))^m.$$

In particular, the above identity holds for all $f \in C_c^\infty(\omega(s))^m$ and $g \in C_c^\infty(\omega(t))^m$. Therefore, by the estimate (5.30) and duality, we get

$$(5.31) \quad \|P_{s \rightarrow t}^\phi f\|_{L_2(\omega(t))} \leq N_0 R^{-d/2} e^{KR+\vartheta K^2(t-s)} \|f\|_{L_1(\omega(s))}, \quad \forall f \in C_c^\infty(\omega(s))^m.$$

Now, set $r = (s+t)/2$ and observe that by uniqueness, we have

$$P_{s \rightarrow t}^\phi f = P_{r \rightarrow t}^\phi (P_{s \rightarrow r}^\phi f), \quad \forall f \in C_c^\infty(\omega(s))^m.$$

Then, by noting that $t-r = r-s = (t-s)/2$ and $R/\sqrt{2} \leq \sqrt{t-r} \wedge R_{\max} \leq R$, we obtain from (5.28) and (5.31) that

$$\|P_{s \rightarrow t}^\phi f\|_{L_\infty(\omega(t))} \leq N R^{-d} e^{2KR+\vartheta K^2(t-s)} \|f\|_{L_1(\omega(s))}, \quad \forall f \in C_c^\infty(\omega(s))^m; \quad N = 2^{d/2} N_0^2.$$

For all $x \in \omega(t)$ and $y \in \omega(s)$, the above estimate and (5.26) yield, by duality, that

$$(5.32) \quad e^{\phi(x)-\phi(y)} |\mathbf{G}(t, x, s, y)| \leq NR^{-d} e^{2KR+\vartheta K^2(t-s)}.$$

Let $\phi(z) = K\phi_0(|z - y|)$, where ϕ_0 is defined on $[0, \infty)$ by

$$\phi_0(r) = \begin{cases} r & \text{if } r \leq |x - y| \\ |x - y| & \text{if } r > |x - y|. \end{cases}$$

Then, ϕ is a bounded Lipschitz function on \mathbb{R}^d satisfying $|D\phi| \leq K$ a.e. We set

$$K = |x - y|/2\vartheta(t - s) \quad \text{and} \quad \xi := |x - y|/\sqrt{t - s}.$$

By (5.32) and the obvious inequality $R/\sqrt{t - s} \leq 1$, we have

$$|\mathbf{G}(t, x, s, y)| \leq NR^{-d} \exp\{\xi/\vartheta - \xi^2/4\vartheta\}.$$

Let $N = N(\vartheta) = N(\nu)$ be chosen so that

$$\exp(\xi/\vartheta - \xi^2/4\vartheta) \leq N \exp(-\xi^2/8\vartheta), \quad \forall \xi \in [0, \infty).$$

If we set $\kappa = 1/8\vartheta = \nu^3/8$, then we obtain

$$|\mathbf{G}(t, x, s, y)| \leq NR^{-d} \exp\{-\kappa|x - y|^2/(t - s)\}$$

where $N = N(d, m, \nu, N_0) > 0$ and recall that we set $R = \sqrt{t - s} \wedge R_{\max}$. The proof is complete. \blacksquare

5.3. Proof of Theorem 3.16. Notice that by Lemma 6.1 and Theorem 3.11, for all $X = (t, x)$ and $Y = (y, s)$ in Ω with $t > s$ we have

$$(5.33) \quad |\mathbf{G}(t, x, s, y)| \leq C_1 \left\{ (t - s) \wedge R_{\max}^2 \right\}^{-d/2} \exp\{-\kappa|x - y|^2/(t - s)\},$$

where $C_1 = C_1(n, m, \nu, \mu_0, N_1)$. We denote

$$\delta_1(X, Y) = \left(1 \wedge \frac{d^-(X)}{R_{\max} \wedge |X - Y|_{\mathcal{D}}} \right) \quad \text{and} \quad \delta_2(X, Y) = \left(1 \wedge \frac{d^+(Y)}{R_{\max} \wedge |X - Y|_{\mathcal{D}}} \right)$$

so that $\delta(X, Y) = \delta_1(X, Y) \delta_2(X, Y)$. To prove the estimate (3.17), we first claim that

$$(5.34) \quad |\mathbf{G}(t, x, s, y)| \leq N \delta_1(X, Y)^{\mu_0} \left\{ (t - s) \wedge R_{\max}^2 \right\}^{-d/2} \exp\{-\kappa|x - y|^2/4(t - s)\},$$

where $N = N(n, m, \nu, \mu_0, N_1)$. The following lemma is a key to prove the above claim.

Lemma 5.35. *For $R \in (0, R_{\max})$ and $X \in \Omega$ such that $d^-(X) < R/2$, let \mathbf{u} be a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $\Omega_-[X, R]$ vanishing on $\mathcal{P}\Omega_-[X, R]$. Then, we have*

$$(5.36) \quad |\mathbf{u}(X)| \leq Nd^-(X)^{\mu_0} R^{-d/2-1-\mu_0} \|\mathbf{u}\|_{L_2(\Omega_-[X, R])},$$

where $N = N(d, m, \nu, \mu_0, N_1, M)$.

Proof. By the very definition the condition (LH), we have

$$(5.37) \quad |\tilde{\mathbf{u}}(Y) - \tilde{\mathbf{u}}(X)| \leq N|Y - X|_{\mathcal{D}}^{\mu_0} R^{-d/2-1-\mu_0} \|\mathbf{u}\|_{L_2(\Omega_-[X, R])}, \quad \forall Y \in Q_-(X, R/2).$$

For any r satisfying $d^-(X) < r < R/2$, there is $Y \in Q_-(X, R/2) \setminus \Omega$ such that $|X - Y|_{\mathcal{D}} = r$. By (5.37) we obtain

$$|\mathbf{u}(X)| = |\tilde{\mathbf{u}}(X) - \tilde{\mathbf{u}}(Y)| \leq Nr^{\mu_0} R^{-d/2-1-\mu_0} \|\mathbf{u}\|_{L_2(\Omega_-[X, R])}.$$

By taking limit $r \rightarrow d^-(X)$ in the above inequality, we derive (5.36). \blacksquare

Now we are ready to prove (5.34). Take $R = (R_{\max} \wedge |X - Y|_{\mathcal{P}})/4$. We may assume that $d^-(X) < R/2$ because otherwise (5.34) follows from (5.33). We then set \mathbf{u} to be the columns of $\mathbf{G}(\cdot, Y)$ in Lemma 5.35 to obtain

$$(5.38) \quad |\mathbf{G}(X, Y)| \leq Cd^-(X)^{\mu_0} R^{-d/2-1-\mu_0} \|\mathbf{G}(\cdot, Y)\|_{L_2(\Omega_-[X, R])}, \quad R = (R_{\max} \wedge |X - Y|_{\mathcal{P}})/4.$$

Next, we consider the following three possible cases.

Case 1: $|x - y| \leq \sqrt{t - s} < R_{\max}$. In this case $R = \sqrt{t - s}/4 = |X - Y|_{\mathcal{P}}/4$ and thus, we get from (5.38) and (3.14) that

$$|\mathbf{G}(X, Y)| \leq Nd^-(X)^{\mu_0} R^{-d/2-1-\mu_0} \|\mathbf{G}(\cdot, Y)\|_{L_2(\Omega_-[X, R])} \leq Nd^-(X)^{\mu_0} R^{-d-\mu_0},$$

which immediately implies (5.34) in this case.

Case 2: $\sqrt{t - s} < |x - y| \wedge R_{\max}$. In this case $R = (|x - y| \wedge R_{\max})/4$. We denote $Z = (r, z)$ and claim that for all $Z \in \Omega_-(X, 2R)$, we have

$$(5.39) \quad |\mathbf{G}(r, z, s, y)| \leq NC_1(t - s)^{-d/2} \exp\{-\kappa|x - y|^2/4(t - s)\},$$

where C_1 and κ are the same constants as in (5.33) and $N = N(d, \kappa)$. To prove the claim, first note that we may assume $Y = 0$ without loss of generality. Then by (5.33) we have

$$|\mathbf{G}(r, z, s, y)| \leq C_1 r^{-d/2} e^{-\kappa|z|^2/r} \chi_{(0, \infty)}(r) \leq C_1 r^{-d/2} e^{-\kappa|x|^2/4r} \chi_{(0, \infty)}(r),$$

where we used $|z| = |z - y| \geq |x - y|/2 = |x|/2$. Let us denote

$$g(\tau) = \tau^{-d/2} e^{-\kappa|x|^2/4\tau} \chi_{(0, \infty)}(\tau), \quad g_0(\tau) = \tau^{-d/2} e^{-\kappa/4\tau} \chi_{(0, \infty)}(\tau).$$

Then the claim (5.39) will follow if we show that there exists a positive number $N = N(d, \kappa)$ such that $g(r) < Ng(t)$ for all $r < t < |x|^2$, which in turn will follow if we show that $g_0(r_1) \leq Ng_0(r_2)$ for all $r_1 < r_2 \leq 1$. But the latter assertion is easy to verify by an elementary analysis of the function g_0 .

We have thus proved (5.39), which combined with (5.38) yields

$$|\mathbf{G}(X, Y)| \leq Cd^-(X)^{\mu_0} R^{-\mu_0} (t - s)^{-d/2} \exp\{-\kappa|x - y|^2/4(t - s)\}.$$

Therefore, we also obtain (5.34) in this case.

Case 3: $R_{\max} \leq \sqrt{t - s}$. In this case $R = R_{\max}/4$, and the desired estimate (5.34) becomes

$$(5.40) \quad |\mathbf{G}(t, x, s, y)| \leq C\{d^-(X)/R_{\max}\}^{\mu_0} R_{\max}^{-d} \exp\{-\kappa|x - y|^2/4(t - s)\}.$$

Since $t - s \geq 16R^2$, for all $Z = (r, z) \in \Omega_-(X, 2R)$, we have

$$(5.41) \quad \exp\left\{-\kappa \frac{|z - y|^2}{r - s}\right\} \leq \exp\left\{-\kappa \frac{|x - y|^2/2 - |z - x|^2}{t - s}\right\} \leq e^{\kappa/4} \exp\left\{-\frac{\kappa|x - y|^2}{2(t - s)}\right\}.$$

Then, from (5.38), (5.33), and (5.41), we obtain (5.40), which implies (5.34) in this case.

We have thus proved that the estimate (5.34) holds in all possible cases. Finally, notice that Lemma 5.35 remains valid if $\mathcal{L}, X, d^-(X), \Omega_-[X, R]$, and $\mathcal{P}\Omega_-[X, R]$, respectively, are replaced by ${}^t\mathcal{L}, Y, d^+(Y), \Omega_+[Y, R]$, and $\mathcal{P}\Omega_+[Y, R]$. Therefore, by replicating the above argument to $\tilde{\mathbf{G}}(\cdot, X)$, utilizing the estimate (5.34) instead of (5.33), and using the identity (3.4), we obtain

$$|\mathbf{G}(t, x, s, y)| \leq N\delta_2(X, Y)^{\mu_0} \delta_1(X, Y)^{\mu_0} \{(t - s) \wedge R_{\max}^2\}^{-d/2} \exp\{-\kappa|x - y|^2/16(t - s)\}.$$

By replacing κ by $\kappa/16$, we obtain the desired estimate (3.17). The theorem is proved. \blacksquare

6. APPENDIX

Lemma 6.1. *Assume the condition (LH). Then the condition (LB) is satisfied with the same R_{max} and $N_0 = N_0(n, m, \mu_0, N_1)$.*

Proof. Let \mathbf{u} be a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $\Omega_-[X, R]$ vanishing on $\mathcal{P}\Omega_-[X, R]$, where $X \in \Omega$ and $R \in (0, R_{max})$. By using the triangle inequality, for all $Y \in Q_-(X, R/2)$ and $Z \in Q_-(Y, R/2)$, we have

$$|\tilde{\mathbf{u}}(Y)|^2 \leq 2|\tilde{\mathbf{u}}(Y) - \tilde{\mathbf{u}}(Z)|^2 + 2|\tilde{\mathbf{u}}(Z)|^2 \leq NR^{2\mu_0}[\tilde{\mathbf{u}}]_{\mu_0/2, \mu_0; Q_-(X, R)}^2 + N|\tilde{\mathbf{u}}(Z)|^2.$$

Then by taking average over $Z \in Q_-(Y, R/2)$ and using (LH), we obtain

$$\|\mathbf{u}\|_{L_\infty(\Omega_-[X, R/2])}^2 \leq NR^{2\mu_0}[\tilde{\mathbf{u}}]_{\mu_0/2, \mu_0; Q_-(X, R)}^2 + NR^{-d-2}\|\tilde{\mathbf{u}}\|_{L_2(Q_-(X, R))}^2 \leq NR^{-d-2}\|\mathbf{u}\|_{L_2(\Omega_-[X, R])}^2,$$

where $N = N(d, m, \mu_0, N_1)$. This proves the part i) of the condition (LB). The proof for the other part is very similar and is omitted. \blacksquare

Lemma 6.2. *Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . Assume that \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = \mathbf{f}$ in $\Omega_-[X_0, R]$ vanishing on $\mathcal{P}\Omega_-[X_0, R]$, where $\mathbf{f} \in L_\infty(\Omega_-[X_0, R])$, and denote $\tilde{\mathbf{u}} = \chi_{\Omega_-[X_0, R]}\mathbf{u}$. Then we have*

$$(6.3) \quad \int_{Q_-(X_0, R)} |\tilde{\mathbf{u}} - (\tilde{\mathbf{u}})_{X_0, R}|^2 \leq NR^2 \int_{\Omega_-[X_0, R]} |D\mathbf{u}|^2 + NR^{2-d}\|\mathbf{f}\|_{L_1(\Omega_-[X_0, R])}^2,$$

where $(\tilde{\mathbf{u}})_{X_0, R} = \oint_{Q_-(X_0, R)} \tilde{\mathbf{u}}$ and $N = N(d, m, \nu)$.

Proof. We modify the proof of [30, Lemma 3]. Without loss of generality, we may assume $X_0 = 0$. Let $\zeta = \zeta(x)$ be a smooth function defined on \mathbb{R}^d such that

$$0 \leq \zeta \leq 1, \quad \text{supp } \zeta \subset B(R), \quad \zeta \equiv 1 \text{ on } B(R/2), \quad \text{and} \quad |D\zeta| \leq 4/R.$$

Setting $\delta^{-1} = \int_{B(R)} \zeta(x) dx$ and denote

$$\beta(t) := \delta \int_{B(R)} \zeta(x) \tilde{\mathbf{u}}(t, x) dx = \delta \int_{\omega(t) \cap B(R)} \zeta(x) \mathbf{u}(t, x) dx, \quad \bar{\beta} := R^{-2} \int_{-R^2}^0 \beta(t) dt.$$

Since $(\tilde{\mathbf{u}})_{0, R}$ minimizes the integral $\int_{Q_-(R)} |\tilde{\mathbf{u}} - \mathbf{c}|^2$ among $\mathbf{c} \in \mathbb{R}^m$, we obtain

$$(6.4) \quad \int_{Q_-(R)} |\tilde{\mathbf{u}} - (\tilde{\mathbf{u}})_{0, R}|^2 \leq \int_{Q_-(R)} |\tilde{\mathbf{u}} - \bar{\beta}|^2 \leq 2 \int_{Q_-(R)} |\tilde{\mathbf{u}} - \beta(t)|^2 + 2 \int_{Q_-(R)} |\beta(t) - \bar{\beta}|^2.$$

Notice that $\tilde{\mathbf{u}} \in W_2^{0,1}(Q_-(R))$ and $D\tilde{\mathbf{u}} = \chi_{\Omega_-[R]} D\mathbf{u}$ in $Q_-(R)$. Therefore, by a variant of Poincaré's inequality, we have

$$(6.5) \quad \int_{Q_-(R)} |\tilde{\mathbf{u}} - \beta(t)|^2 dX = \int_{-R^2}^0 \int_{B(R)} |\tilde{\mathbf{u}}(t, x) - \beta(t)|^2 dx dt \leq N \int_{\Omega_-[R]} |D\mathbf{u}|^2 dX.$$

We claim that for all s and t satisfying $-R^2 < s < t < 0$, we have

$$(6.6) \quad |\beta(t) - \beta(s)|^2 \leq NR^{-d} \int_{\Omega_-[R]} |D\mathbf{u}|^2 + NR^{-2d}\|\mathbf{f}\|_{L_1(\Omega_-[R])}^2.$$

Assume the estimate (6.6) for the moment. By the definition of $\bar{\beta}$, we then obtain

$$(6.7) \quad \begin{aligned} \int_{Q_-(R)} |\beta(t) - \bar{\beta}|^2 dX &= |B(R)| \int_{-R^2}^0 |\beta(t) - \bar{\beta}|^2 dt \\ &\leq NR^{d-2} \int_{-R^2}^0 \int_{-R^2}^0 |\beta(t) - \beta(s)|^2 ds dt \leq NR^2 \int_{\Omega_-[R]} |D\mathbf{u}|^2 dX + NR^{2-d}\|\mathbf{f}\|_{L_1(\Omega_-[R])}^2. \end{aligned}$$

By combining (6.4), (6.5), and (6.7), we obtain (6.3).

It remains us to prove the estimate (6.6). Setting $\boldsymbol{\eta} = \boldsymbol{\eta}(x) = \zeta(x)\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda} \in \mathbb{R}^m$ is a constant column vector, and following the calculation in Brown et al. [3], we obtain

$$\begin{aligned} \int_{\omega(t) \cap B(R)} \zeta(x) \mathbf{u}(t, x) \cdot \boldsymbol{\Lambda} dx - \int_{\omega(s) \cap B(R)} \zeta(x) \mathbf{u}(s, x) \cdot \boldsymbol{\Lambda} dx \\ + \int_{\Omega(s, t)} \boldsymbol{\Lambda}^T \mathbf{A}^{\alpha\beta}(X) D_\beta \mathbf{u}(X) D_\alpha \zeta(x) dX = \int_{\Omega(s, t)} \mathbf{f}(X) \cdot \boldsymbol{\Lambda} \zeta(x) dX. \end{aligned}$$

Notice that $\delta^{-1} \geq 2^{-d}|B(1)|R^d$. Therefore, by using the properties of the function ζ , we get for all s and t satisfying $-R^2 < s < t < 0$ that

$$(\boldsymbol{\beta}(t) - \boldsymbol{\beta}(s)) \cdot \boldsymbol{\Lambda} \leq NR^{-d-1}|\boldsymbol{\Lambda}| \int_{\Omega_-[R]} |D\mathbf{u}| dX + NR^{-d}|\boldsymbol{\Lambda}| \int_{\Omega_-[R]} |\mathbf{f}| dX.$$

By taking $\boldsymbol{\Lambda} = \boldsymbol{\beta}(t) - \boldsymbol{\beta}(s)$ in the above inequality and using Hölder's inequality and Cauchy's inequality with ε , we obtain (6.6). The proof is complete. \blacksquare

Lemma 6.8. *Let Ω be a time-varying H_1 (graph) domain in \mathbb{R}^{d+1} . Let $a^{\alpha\beta}$ satisfy (4.5) and let \mathcal{E} be as in (4.6), where $A_{ij}^{\alpha\beta}$ are the coefficients of the operator \mathcal{L} . Then, there exists $\mathcal{E}_0 = \mathcal{E}_0(d, \nu_0, M) > 0$ such that if $\mathcal{E} < \mathcal{E}_0$, then the condition (LH) is satisfied with $\mu_0 = \mu_0(d, \nu_0, M)$, $R_{\max} = R_a$, and $N_1 = N_1(d, m, \nu_0, M)$. Here, we set $R_a = \infty$ if Ω is a time-varying H_1 graph domain.*

Proof. We shall prove below that there exists a number $\mathcal{E}_1 = \mathcal{E}_1(d, \nu_0, M) > 0$ such that if $\mathcal{E} < \mathcal{E}_1$, then the following holds: There exist positive constants $\mu_1 = \mu_1(d, \nu_0, M)$ and $C_1 = C_1(d, m, \nu_0, M)$ such that for any $\tilde{X} \in \partial\Omega = \mathcal{P}\Omega$ and $R \in (0, R_a)$, if \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $\Omega_-[\tilde{X}, R]$ vanishing on $\mathcal{P}\Omega_-[\tilde{X}, R]$, then we have

$$(6.9) \quad \int_{\Omega_-[\tilde{X}, \rho]} |D\mathbf{u}|^2 \leq C_1 \left(\frac{\rho}{r}\right)^{d+2\mu_1} \int_{\Omega_-[\tilde{X}, r]} |D\mathbf{u}|^2, \quad \forall 0 < \rho < r \leq R.$$

We also note that by [4, Lemma 2.2], there is $\mathcal{E}_2 = \mathcal{E}_2(d, \nu_0) > 0$ such that if $\mathcal{E} < \mathcal{E}_2$, then the following holds: There exists a constant $\mu_2 = \mu_2(d, \nu_0) \in (0, 1]$ such that if \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $Q_-(X, R) \subset \Omega$, then we have

$$(6.10) \quad \int_{Q_-(X, \rho)} |D\mathbf{u}|^2 \leq C_2 \left(\frac{\rho}{r}\right)^{d+2\mu_2} \int_{Q_-(X, r)} |D\mathbf{u}|^2, \quad \forall 0 < \rho < r \leq R,$$

where $C_2 = C_2(d, m, \nu_0)$. Then we combine (6.9) and (6.10), via a standard method in boundary regularity theory to conclude that if $\mathcal{E} < \mathcal{E}_1 \wedge \mathcal{E}_2 =: \mathcal{E}_0$, then for all $X \in \Omega$ and $0 < R < R_a$, the following holds: If \mathbf{u} is a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $\Omega_-[X, R]$ vanishing on $S\Omega_-[X, R]$, then we have

$$(6.11) \quad \int_{\Omega_-[X, \rho]} |D\mathbf{u}|^2 \leq N \left(\frac{\rho}{r}\right)^{d+2\mu_0} \int_{\Omega_-[X, r]} |D\mathbf{u}|^2, \quad \forall 0 < \rho < r \leq R,$$

where $\mu_0 = \mu_1 \wedge \mu_2$ and $N = N(d, m, \nu_0, M)$. By Lemma 6.2, the estimate (6.11), and the energy inequality of Brown et. al [3], we have for all $Y \in Q_-(X, R/4)$ and $r \in (0, R/4)$ that

$$\begin{aligned} \int_{Q_-(Y, r)} |\tilde{\mathbf{u}} - (\tilde{\mathbf{u}})_{Y, r}|^2 &\leq Nr^2 \int_{\Omega_-[Y, r]} |D\mathbf{u}|^2 \leq Nr^2 \left(\frac{r}{R}\right)^{d+2\mu_0} \int_{\Omega_-[Y, R/4]} |D\mathbf{u}|^2 \\ &\leq N \left(\frac{r}{R}\right)^{d+2+2\mu_0} \int_{\Omega_-[Y, R/2]} |\mathbf{u}|^2 \leq Nr^{d+2+2\mu_0} R^{-2\mu_0} \int_{Q_-(X, R)} |\tilde{\mathbf{u}}|^2, \end{aligned}$$

where $N = N(d, m, \nu_0, M)$. Then by the Campanato's characterization of Hölder continuous functions (see e.g., [4, Lemma 2.5]), we obtain

$$[\tilde{\mathbf{u}}]_{\mu_0/2, \mu_0; Q_-(X, R/4)}^2 \leq CR^{-2\mu_0} \int_{Q_-(X, R)} |\tilde{\mathbf{u}}|^2.$$

Then, the above inequality together with a standard covering argument yields part i) of the condition (LH). The other part of the condition (LH) is similarly obtained.

Now, it only remains for us to prove the estimate (6.9). For $\tilde{X} \in \partial\Omega$ and $R \in (0, R_a)$ given, let \mathbf{u} be a weak solution of $\mathcal{L}\mathbf{u} = 0$ in $\Omega_-[\tilde{X}, R]$ vanishing on $\mathcal{P}\Omega_-[\tilde{X}, R]$. Denote by \mathcal{L}_0 the parabolic operator acting on scalar functions v as follows:

$$\mathcal{L}_0 v = v_t - D_\alpha(a^{\alpha\beta} D_\beta v).$$

For $r \in (0, R]$, let v^i be a unique weak solution in $V_2(\Omega_-[\tilde{X}, r])$ of the problem

$$\begin{cases} \mathcal{L}_0 v^i = 0 & \text{in } \Omega_-[\tilde{X}, r], \\ v^i = u^i & \text{on } \mathcal{P}(\Omega_-[\tilde{X}, r]), \end{cases}$$

where $i = 1, \dots, m$. Existence of such v^i follows from Brown et al. [3]. We claim that there are positive constants $\mu = \mu(d, \nu_0, M)$ and $N = N(d, m, \nu_0, M)$ such that the following estimate holds:

$$(6.12) \quad \int_{\Omega_-[\tilde{X}, \rho]} |Dv^i|^2 \leq N \left(\frac{\rho}{r} \right)^{d+2\mu} \int_{\Omega_-[\tilde{X}, r]} |Dv^i|^2, \quad \forall 0 < \rho < r.$$

We may assume that $\rho < r/8$ because otherwise (6.12) becomes trivial. Since each v^i vanishes on $\mathcal{P}\Omega_-[\tilde{X}, r]$, it follows from [22, Theorem 6.32] and [22, Theorem 6.30] that there exist $\mu = \mu(d, \nu_0, M) > 0$ and $N = N(d, \nu_0, M) > 0$ such that

$$(6.13) \quad \operatorname{osc}_{\Omega_-[\tilde{X}, 2\rho]} v^i \leq N \rho^\mu r^{-\mu} \sup_{\Omega_-[\tilde{X}, r/4]} |v^i| \leq N \rho^\mu r^{-\mu-d/2-1} \|v^i\|_{L_2(\Omega_-[\tilde{X}, r/2])}.$$

In particular, the estimate (6.13) implies $v^i(\tilde{X}) = 0$. Then, by the energy inequality of Brown et al. [3] and [14, Lemma 4.2], we obtain (recall that $\rho < r/8$)

$$\begin{aligned} \int_{\Omega_-[\tilde{X}, \rho]} |Dv^i|^2 &\leq N \rho^{-2} \int_{\Omega_-[\tilde{X}, 2\rho]} |v^i|^2 = N \rho^{-2} \int_{\Omega_-[\tilde{X}, 2\rho]} |v^i(Y) - v^i(\tilde{X})|^2 dY \\ &\leq N \rho^d \left(\operatorname{osc}_{\Omega_-[\tilde{X}, 2\rho]} v^i \right)^2 \leq N \left(\frac{\rho}{r} \right)^{d+2\mu} r^{-2} \int_{\Omega_-[\tilde{X}, r/2]} |v^i|^2 \\ &\leq N \left(\frac{\rho}{r} \right)^{d+2\mu} \int_{\Omega_-[\tilde{X}, r]} |Dv^i|^2, \quad i = 1, \dots, m. \end{aligned}$$

where $N = N(d, \nu_0, M)$. This completes the proof of the estimate (6.12). Next, notice that $\mathbf{w} := \mathbf{u} - \mathbf{v}$ belongs to $V_2(\Omega_-[\tilde{X}, r])$, vanishes on $\mathcal{P}(\Omega_-[\tilde{X}, r])$, and satisfies weakly

$$\mathcal{L}_0 \mathbf{w} = D_\alpha \left((A^{\alpha\beta} - a^{\alpha\beta} I_m) D_\beta \mathbf{u} \right).$$

Therefore, by the energy inequality of Brown et al. [3], we obtain

$$(6.14) \quad \int_{\Omega_-[\tilde{X}, r]} |D\mathbf{w}|^2 \leq N \mathcal{E}^2 \int_{\Omega_-[\tilde{X}, r]} |D\mathbf{u}|^2,$$

where \mathcal{E} is defined as in (4.6). By combining (6.12) and (6.14), we obtain

$$\int_{\Omega_-[\tilde{X}, \rho]} |D\mathbf{u}|^2 \leq N \left(\frac{\rho}{r} \right)^{d+2\mu} \int_{\Omega_-[\tilde{X}, r]} |D\mathbf{u}|^2 + N \mathcal{E}^2 \int_{\Omega_-[\tilde{X}, r]} |D\mathbf{u}|^2, \quad \forall 0 < \rho < r.$$

Now, choose a number $\mu_1 \in (0, \mu)$. Then, by a well known iteration argument (see, e.g., [17, §III.2]), we find that there exists \mathcal{E}_1 such that if $\mathcal{E} < \mathcal{E}_1$, then we have the estimate (6.9). The lemma is proved. ■

Acknowledgment. We thank Steve Hofmann for valuable comments. Hongjie Dong was partially supported by the National Science Foundation under agreement No. DMS-0800129. Seick Kim was supported by supported by Mid-career Researcher Program through NRF grant funded by the MEST (No. 2010-0027491) and also WCU(World Class University) program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (R31-2008-000-10049-0).

REFERENCES

- [1] Aronson, D. G. *Bounds for the fundamental solution of a parabolic equation*. Bull. Amer. Math. Soc. **73** (1967), 890–896.
- [2] Auscher, P. *Regularity theorems and heat kernel for elliptic operators*. J. London Math. Soc. (2) **54** (1996), no. 2, 284–296.
- [3] Brown, R. M.; Hu, W.; Lieberman, G. M. *Weak solutions of parabolic equations in non-cylindrical domains*. Proc. Amer. Math. Soc. **125** (1997), no. 6, 1785–1792.
- [4] Cho, S.; Dong, H.; Kim, S. *On the Green's matrices of strongly parabolic systems of second order*. Indiana Univ. Math. J. **57** (2008), no. 4, 1633–1677.
- [5] Cho, S.; Dong, H.; Kim, S. *Global estimates for Green's matrix of second order parabolic systems with application to elliptic systems in two dimensional domains*. arXiv:1007.5429v1 [math.AP]
- [6] Davies, E. B. *Explicit constants for Gaussian upper bounds on heat kernels*. Amer. J. Math. **109** (1987), no. 2, 319–333.
- [7] Dolzmann, G.; Müller, S. *Estimates for Green's matrices of elliptic systems by L^p theory*. Manuscripta Math. **88** (1995), no. 2, 261–273.
- [8] Dong, H.; Kim, S. *Green's matrices of second order elliptic systems with measurable coefficients in two dimensional domains*. Trans. Amer. Math. Soc. **361** (2009), 3303–3323.
- [9] Fabes, E. B.; Stroock, D. W. *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*. Arch. Rational Mech. Anal. **96** (1986), no. 4, 327–338.
- [10] Fuchs, M. *The Green-matrix for elliptic systems which satisfy the Legendre-Hadamard condition*. Manuscripta Math. **46** (1984), no. 1-3, 97–115.
- [11] Fuchs, M. *The Green matrix for strongly elliptic systems of second order with continuous coefficients*. Z. Anal. Anwendungen **5** (1986), no. 6, 507–531.
- [12] Grüter, M.; Widman, K.-O. *The Green function for uniformly elliptic equations*. Manuscripta Math. **37** (1982), no. 3, 303–342.
- [13] Hofmann, S.; Kim, S. *Gaussian estimates for fundamental solutions to certain parabolic systems*. Publ. Mat. **48** (2004), no. 2, 481–496.
- [14] Hofmann, S.; Kim, S. *The Green function estimates for strongly elliptic systems of second order*. Manuscripta Math. **124** (2007), no. 2, 139–172.
- [15] Hofmann S.; Lewis, J. L. *L^2 solvability and representation by caloric layer potentials in time-varying domains*. Ann. of Math. (2) **144** (1996), no. 2, 349–420.
- [16] Hofmann, S.; Nyström, K. *Dirichlet problems for a nonstationary linearized system of Navier-Stokes equations in non-cylindrical domains*. Methods Appl. Anal. **9** (2002), no. 1, 13–98.
- [17] Giaquinta, M. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*. Princeton University Press:Princeton, NJ, 1983.
- [18] Kim, S. *Gaussian estimates for fundamental solutions of second order parabolic systems with time-independent coefficients*. Trans. Amer. Math. Soc. **360** (2008),. 6031–6043.
- [19] Krylov N. V. *Parabolic and elliptic equations with VMO coefficients*. Comm. Partial Differential Equations, **32** (2007), no. 3, 453–475.
- [20] Ladyzhenskaya, O. A.; Solonnikov, V. A.; Ural'tseva, N. N. *Linear and quasilinear equations of parabolic type*. American Mathematical Society: Providence, RI, 1967.
- [21] Lewis, J. L.; Murray, M. A. M. *The method of layer potentials for the heat equation in time-varying domains*. Mem. Amer. Math. Soc. **114** (1995), no. 545.
- [22] Lieberman G. M. *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

- [23] Littman, W.; Stampacchia, G.; Weinberger, H. F. *Regular points for elliptic equations with discontinuous coefficients*. Ann. Scuola Norm. Sup. Pisa (3) **17** (1963) 43–77.
- [24] Moser, J. *A Harnack inequality for parabolic differential equations*. Comm. Pure Appl. Math. **17** (1964), 101–134.
- [25] Nash, J. *Continuity of solutions of parabolic and elliptic equations*. Amer. J. Math. **80** (1958), 931–954.
- [26] Nyström, K. *The Dirichlet problem for second order parabolic operators*. Indiana Univ. Math. J. **46** (1997), no. 1, 183–245.
- [27] Nyström, K. *On area integral estimates for solutions to parabolic systems in time-varying and non-smooth cylinders*. Trans. Amer. Math. Soc. **360** (2008), no. 6, 2987–3017.
- [28] Porper, F. O.; Eidel'man, S. D. *Two-sided estimates of the fundamental solutions of second-order parabolic equations and some applications of them*. (Russian) Uspekhi Mat. Nauk **39** (1984), no. 3(237), 107–156; English translation: Russian Math. Surveys **39** (1984), no. 3, 119–179.
- [29] Rivera-Noriega, J. *Absolute continuity of parabolic measure and area integral estimates in non-cylindrical domains*. Indiana Univ. Math. J. **52** (2003), no. 2, 477–525.
- [30] Struwe, M. *On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems*. Manuscripta Math. **35** (1981), no. 1-2, 125–145.

(H. Dong) DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, 182 GEORGE STREET, BOX F, PROVIDENCE, RI 02912, USA

E-mail address: Hongjie.Dong@brown.edu

(S. Kim) DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120-749, REPUBLIC OF KOREA

Current address: Department of Computational Science and Engineering, Yonsei University, Seoul 120-749, Republic of Korea

E-mail address: kimseick@yonsei.ac.kr